Notes on Cosmology

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1 Cosmological Principle

Now we leave behind galaxies and beginning cosmology.

Cosmology is the study of the Universe as a whole. It concerns topics such as the basic content of the universe, the basic parameter such as the density, expansion rate of the universe, and how we measure them, the history of the universe, how it started, how old it is, and the large scale structure of the universe, meaning the large scale distribution of matter in the universe. One of the basic tools in cosmology is to use galaxies as test particles to probe the universe: galaxies give us tools to measure distance, and therefore expansion of the universe, and they are generally under the influence of gravity, so they are excellent probes of how matter distributed and evolve under gravity. So many of the things we learned in the previous part of our class are very useful to cosmology as well. And since galaxies are such excellent probes of cosmology, galaxy formation and evolution are generally viewed as at least part of cosmology as well. In the next 1.5 months, we will first lay down some basic tools of cosmology, then discuss how to test these basic principles, and how to measure basic cosmological parameters, then talk more about dark matter and dark energy, and then spend some time talking about galaxy evolution.

The cosmological principle says: We are not located at any special location in the Universe. The other way to put it is that the universe is homogeneous and isotropic.

Isotropy. This is not so obvious in the nearby galaxies, but distant galaxies, radio sources, and the CMB all show isotropy.

We could be living at the center of a spherically symmetric universe with a radial profile. However, we probably are not at a special point. If two observers both think that the universe is isotropic, then it is in fact homogeneous.

Homogeneity is a stronger result, because it means that we can apply our physical laws from Earth.

Both of these ideas only apply on large scales, hundreds of Mpc. As we will see, the universe is several Gpc across, so there are many patches in the universe with which to define homogeneity.

For the next while, we will consider the universe to be straightly isotropic and homogenous to study the basic principles of cosmology. We will then try to relax this, since, as you know, if the universe is really 100% homogenous, then there will be no gravitational fluctuation and no grow of fluctuation, and therefore no formation of galaxies etc. So in later part, we will consider how the universe is different from completely homogenous, and how these fluctuations grow with time, from the very small fluctuations in CMB to forming large galaxies and large scale structure of galaxies we see today.

2 Metric

The next key concept of modern cosmology is metric, which specifies the distance between two points, in space or in 4-d space time. In 1935, Robertson and Walker were the first to derive (independently) the form of the metric of space-time for all isotropic homogeneous, uniformly expanding model of the Universe. The form of the metric is independent of the assumption that the large-scale dynamics of the universe are described by GR. Whatever form of the physics of expansion, the space-time metric must be of RW form, because, as we will see now, the only assumptions are isotropy and homogeneity. Now let's see how to derive it.

Distant galaxies are observed to be redshifted, as if they are moving away from us. The recessional velocity is proportional to the distance of the galaxy. This is known as the Hubble law. This redshifting is easy to understand in a homogeneous universe. All observers would agree on the expansion and the Hubble constant. However, the fact that the universe used to be smaller does cause some computational complications in the coordinate system. We won't be deriving cosmology from general relativity in this course, but we do need to use these coordinate systems.

Key concept is the metric, which specifies the distance between two points (in space or space-time).

Imagine a 2-dim flat space. We are used to measuring the distance between 2 points by the Pythagorean theorem.

$$(\Delta\ell)^2 = (\Delta x)^2 + (\Delta y)^2$$

If the two points are infinitesimally close together, then we can write the distance as

$$d\ell^2 = dx^2 + dy^2$$

However, the idea of differencing the coordinates only works in Cartesian coordinates. If we switch to polar coordinates, then the distance between two neighboring points is

$$d\ell^2 = dr^2 + r^2 d\theta^2$$

The formula for measuring the distance is called the metric. In general, there could be mixed terms, e.g. $dr\, d\theta$, but we won't need these much. Sometimes, one writes this as a matrix $d\ell^2 = g_{\mu\nu}dx^\mu dx^\nu$, in which case the matrix $g_{\mu\nu}$ is called the metric tensor. Note that the metric can depend on the coordinates themselves.

To compute the length of a curve in our space, we integrate the length $d\ell$ along the curve. We are interested in homogeous space, following the cosmological principles. The plane is an obvious example of a space in which all points are the same. The sphere is another example.

Next, imagine the surface of a sphere of radius R_c. Here, using spherical coords, we have

$$d\ell^2 = R_c^2 d\theta^2 + R_c^2 \sin^2\theta \, d\phi^2$$

Instead of using θ, let's switch variables to the distance from the North pole $r = R_c\theta$. This is a polar coordinate system on the surface of a sphere, a system that people living on the surface of the sphere would choose to use, it would be natural for them to use a polar system, rather than using two angles, which they can't really measure by being on the surface. This makes the metric

$$d\ell^2 = dr^2 + R_c^2 \sin^2(r/R_c)\, d\phi^2$$

This differs from our result with polar coordinates on the flat space. One of the important results from differential geometry is that no change of coordinates can turn one metric into the other. In other words, there is no coordinate system on the sphere that looks like the Cartesian metric. This is because the sphere has a curvature that cannot be done away with by a coordinate transformation. Indeed, this leads one to the result that the metric gives a space its curvature without any reference to the embedding space of the manifold.

However, near any point, one can always define a coordinate system that appears Cartesian. In our case, near the North Pole, if $r \ll R_c$ we will recover the polar coordinate metric, which we know is just a transformation of the Cartesian one.

Our spherical metric yields circles of constant r have smaller circumferences than flat space would imply, $2\pi R_c \sin(r/R_c)$ instead of $2\pi r$.

The opposite case, one in which circles are larger than expected, is also interesting. We write

$$d\ell^2 = dr^2 + R_c^2 \sinh^2(r/R_c)\, d\phi^2$$

Such a metric is known as a hyperbolic geometry. This is similar to the sphere in that one can translate within the manifold and recover the same metric. Unfortunately, this manifold does not have a simple shape in 3-dimensions. The extra circumference makes one imagine a plane crinkled like a saddle, however this is only a local approximation. The key point is that the directions of curvature are in opposite directions. Unlike the sphere, r extends to infinity and the space have infinite area. We will concentrate on the spherical case, the hyperbolic case is completely symmetric to the spherical case, just change sin to sinh.

Now, let's consider 3-dimensional spaces. The flat space metric is easy

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

In spherical coordinates, we have

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Again, the 3-sphere of radius R_c in 4 dimensions has full homogeneity. The metric is

$$d\ell^2 = R_c^2(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2)$$

If we choose $r = R_c\chi$, we reach

$$d\ell^2 = dr^2 + R_c^2 \sin^2(r/R_c) \left[d\theta^2 + \sin^2\theta \, d\phi^2\right]$$

The term in brackets looks like the standard angular distance in spherical coords. Again, as before, if $r \ll R_c$, we recover the flat space metric, whereas at larger r the circumference is smaller than one expects. Again, we have a hyperbolic alternative, with $\sin \rightarrow \sinh$.

These are the only homogeneous 3-dimensional spaces! Of course, one can always change coordinates to disguise the familiar form, but any homogeneous space can be brought to one of these three forms.

We have now generated some coordinate systems for homogeneous spaces. Because we want to discuss time evolution, we must get time into the picture!

In special relativity, we made use of a new length in space-time.

$$ds^2 = dt^2 - \frac{d\ell^2}{c^2}$$

We thought of the motions of particles as curves in space-time in which the time direction was treated specially in the metric.

We want to include the expansion of the universe. When we are including the expansion of the universe, we need to be very careful about the coordinate system. For this, we will choose coordinates that move with the particles. In other words, a given particle that is at rest with respect to the expansion will have constant coordinates. And the expansion of the universe is reflected in the change of the size of the universe as a whole. This coordinate system is called a comoving system, In this system, each galaxy is labelled by r, which is called the comoving radial distance coordinate. It is fixed, i.e., not changing with time. Note that we are dealing with isotropic and homogeneous universe here, so there is no peculiar velocity. The expansion of the universe is defined in a quantity $R(t)$, the scale factor.

In these coordinates, the spatial metric is

$$d\ell^2 = R(t)^2 \left[dr^2 + R_c^2 \sin^2(r/R_c) \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

where $R(t)$ will simply change all the distances by some function of time. Note that R_c is time-independent. It is conventional to pick $R(t) = 1$ at the present day, so the coordinates reflect the present-day scale of the universe. The full space-time metric is then

$$ds^2 = dt^2 - \frac{R(t)^2}{c^2} \left[dr^2 + R_c^2 \sin^2(r/R_c) \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

This is known as the Robertson-Walker metric. It is also common for $R(t)$ to be written $a(t)$. $a(t)$ is called the expansion factor.

This is a very important formula! From this, one calculates all distances and volumes in cosmology. Note that while can compute between any two points, it is often simplest to put one of the points at the origin $r = 0$. This avoids the problem of solving for the equation of straight line.

3 Observations in Cosmology

3.1 Redshift

First, consider the propagation of photons. For photons, $ds = 0$. We will place ourselves at the origin $r = 0$ and consider the arrival of photons from a source at location r_1. Obviously, the photon travels purely radially in our coordinate system, so $d\theta = d\phi = 0$. We have

$$\frac{c}{R(t)} dt = -dr$$

where the minus sign is because the photon is traveling toward the origin.

Let's imagine two signals sent to us. One leaves at t_1 and is received at t_0; the next is sent at $t_1 + \Delta t_1$ and is received at $t_0 + \Delta t_0$. We want to find out the relation between Δt_0 and Δt_1.

Let's integrate along the path to find the travel time.

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

The second signal is

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

Note that the coordinate distance traveled is the same; we have put the expansion of the universe in the $R(t)$ term, while the source and observer sit at fixed r.

Setting these equal gives

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c}{R(t)} dt = \int_{t_1}^{t_0} \frac{c}{R(t)} dt + \frac{c\Delta t_0}{R(t_0)} - \frac{c\Delta t_1}{R(t_1)}$$

So

$$\Delta t_0 = \Delta t_1 \frac{R(t_0)}{R(t_1)}$$

The difference of the arrival times has been dilated relative to the difference of the departure times by a factor that is the ratio of the expansion factors at the two times. If we imagine that our departure times were successive crests of an electromagnetic wave, then the period of the light must be altered by a factor $R(t_0)/R(t_1)$. This is the same as the time dilation in special relativity, and can be tested.

The light appears Doppler shifted. Its wavelength is changed by a factor $R(t_0)/R(t_1)$. We normally define the redshift by $\lambda_{obs} = \lambda_{emit}(1 + z)$. This means that

$$1 + z = R(t_0)/R(t_1)$$

Again, the expansion of the universe causes this redshift without reference to gravity. One can think of the wavelength of the light being stretched by the expansion. Light from redshift $z = 1$ was emitted when the universe was half its present size.

Locally, we like to think of the redshift as a velocity effect. The Hubble law is written as a ratio of velocity to distance.

looking the distance:

$$\int_{t_0}^{t_1} \frac{c}{R(t)} dt = -\int_r^0 dr = r,$$

when Δt is small, we have $c\Delta t = R(t_0)r$.

ow for velocity:

$$1 + z = \frac{R(t_0)}{R(t_1)} = \frac{R(t_0)}{1 + \Delta t \dot{R}(t)},$$

$$z = |\Delta t| \frac{\dot{R}(t_0)}{R(t_0)} = |\Delta t| H_0.$$

For small redshifts, the velocity is $v = cz$, we have

$$v = cz = c|\Delta t| H_0 = r H_0,$$

and the ratio of the velocity to the distance $R(t_0)r$ is $H_0 = \dot{R}(t_0)/R(t_0)$. Remember this, it is the definition of Hubble constant in cosmology. Hubble constant is the RATE of expansion of the universe, it is the rate of the increase of the scale factor, or of the size of the universe.

One can define the Hubble constant as a function of time $H(t) = \dot{R}/R$. This relates the observed expansion to the behavior of $R(t)$.

3.2 Distance

What distances do we actually measure in cosmology? We have seen one, the comoving distance traveled in a given time

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

Note that if we change integration variables from t to $z = 1/R(t) - 1$, then we have $dz = -(dR/dt)/R^2 \, dt = -H(t)(1 + z) \, dt$. So

$$r = \int_0^z \frac{c \, dz}{H(t)}$$

We will eventually have simple formulae for $H(z)$, but it's arbitrary for now.

However, we will find that this distance isn't necessarily the one that we measure! In fact, we can't measure r, the comoving distance at all.

Recall that we have been describing a number of ways to measure distance earlier in the semester. How do we measure them? We can't go there and have a ruler. We more or less use a combination of two ways. First, if we know the true angular diameter of something by other means, and we can measure the angular diameter observed using telescope, then we get the distance. Examples are like the definition of parellax, and the moving cluster method. This is called the angular diameter distance, $D_A = d/\Delta\theta$.

Second, if we know the true luminosity of something by other means, and we can measure the apparent luminosity, then we get the distance. Examples are like Cephids, T-F etc. This is called the Luminosity Distance. $f = L/(4\pi D_L^2)$.

Now let's see how to measure these distances in a cosmological context.

Consider an object of some known physical diameter at some redshift. What angle on the sky do we measure?

We will imagine ourselves at $r = 0$ and consider the object to be extended in the θ direction. Its diameter is

$$d = R(t)R_c \sin(r/R_c)\Delta\theta$$

We would expect that $d/\Delta\theta$ would be the distance to the source. We define this ratio as the "angular diameter distance" D_A.

$$D_A = R(t)R_c \sin(r/R_c) = \frac{R_c \sin(r/R_c)}{1+z}$$

We computed $r(z)$ above.

In the flat cosmology, $D_A = r/(1+z)$. Note that the curvature causes the transverse distances to differ from the radial ones!

Next, let's consider a source of some luminosity L at some redshift z. We would expect the flux to be $L/4\pi D^2$. What do we actually get?

Place the source at the origin. How much solid angle does our telescope subtend? The metric says the transverse distance today corresponding to a given angle is $R(t_0)R_c \sin(r/R_c)\Delta\theta$. So the solid angle of an area A is $A[R_c \sin(r/R_c)]^{-2}$.

Is that all? There are other effects, however. The source is emitting a given number of photons per second, but we receive these photons in a time that is stretched by $R(t_0)/R(t_1)$. So the number flux is reduced by $1+z$. Moreover, each photon is redshifted and has its frequency reduced by $1+z$. So the energy flux is reduced by $(1+z)^2$.

We receive a flux

$$f = \frac{L}{4\pi[R_c \sin(r/R_c)]^2(1+z)^2}$$

If we define the "luminosity distance" by $f = L/r\pi D_L^2$, then

$$D_L = R_c \sin(r/R_c)(1+z) = D_A(1+z)^2$$

Warning: if one is concerned with the flux per unit frequency, then $f_\nu \neq L_{\nu(1+z)}/4\pi D_L^2$. The flux over some range in frequency does scale as D_L^{-2}, but the range of frequency itself is scaling as $(1+z)^{-1}$. So $f_\nu = L_{\nu(1+z)}(1+z)/4\pi D_L^2$.

If the flux received goes as $f \sim 1/D_L^2$ while the angle subtended goes as $\Omega \sim 1/D_A^2$, then the surface brightness is going as $SB \sim f/\Omega \sim D_A^2/D_L^2$. This always goes as $(1+z)^{-4}$ without any reference to the density of the universe. High-redshift objects have lower surface brightness.

There are a couple of other distances that can be measured, but they don't occur very often. The luminosity distance and angular diameter distance are the total major ways that we express distance in cosmology. Note that because the expansion of the universe, and the fact that how we define distance depends on the expansion in different ways, there are many different distances in cosmology, and for anything with substantial redshift, say $z > 0.1$, they are different. So when you say something is N Mpc away, it is not meaningful unless you see what it means. Note that all these distance measurements also involve calculating the comoving distance $r = \int cdt/R(t)$, and R_c, the radius of curvature of the universe, which we don't know unless we (1) find out the curvature, and (2) solve the evolution of $R(t)$. A much better way is to write down things in redshift, and if you know the evolution of $R(t)$, then we can calculate distance in any way you want. However, the true importance of talking about these distances is that given redshift, it gives you the relation between size and angular diameter, between apparent and absolute flux. That's what you really need when you are measuring a distant object and try to interpret the data!

3.3 Look-Back Time

Finally, we would like to calculate the look-back time to a given redshift, the time from a certain redshift to the current epoch, which is something we are clearly interested in, for example, to calculate how much time there is for the stars in a galaxy to evolve if it was formed at $z = 5$.

From $dt = -dr\ R(t)/c$, we have

$$t(z) = \int_0^r \frac{dr}{c} R(t) = \int_0^z \frac{dz}{H(t)} \frac{1}{1+z}$$

We have now studied the Robertson-Walker metric for a general $R(t)$. In fact, the behavior of $R(t)$ depends on the gravitational attractions of the homogeneous matter.

4 Newtonian Analogue

Usually, one derives this from General Relativity, but let us first consider a Newtonian toy problem. Consider a homogeneous density distribution in uniform expansion. If we pick a origin and draw a sphere around it, then we might calculate the gravitational force by Gauss's law. This means that the material outside the sphere doesn't contribute. In detail, we haven't proven this (nor can we).

We will let the radius of the sphere move with its boundary particles. The radius of the sphere is denoted $R(t)$. The mass inside it is constant. If the density today is ρ_0 and the radius today is R_0, then the mass is $4\pi\rho_0 R_0^3/3$. The acceleration is then

$$\ddot{R} = -\frac{4\pi}{3} G\rho_0 \frac{R_0^3}{R^2}$$

As the sphere expands, the density must scale as $\rho = \rho_0 R_0^3 R^{-3}$. So we also have

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3} G\rho R$$

This is a second-order ODE for the expansion of our toy universe. We must specify two boundary conditions, which we will take as the radius and Hubble constant today.

$$R|_{t_0} = R_0$$

$$(\dot{R}/R)|_{t_0} = H_0$$

Using a familiar DE trick:

$$\ddot{R} = d\dot{R}/dt = d\dot{R}/dR\dot{R}$$

We can integrate this equation and get:

$$\frac{1}{2}\dot{R}^2 = \frac{4\pi}{3} G\rho_0 \frac{R_0^3}{R} + C/2$$

Here C is a constant of integral. This ODE is the famous Friedmann equation, which provides the evolution of the scale factor. It is the dynamical equation of the evolution of the universe. Think of it as the Newton's law in cosmology. In fact, we just derived from Newton's law in a homogeneous sphere case. It can be derived from GR, Einstein's field equation, assuming cosmological principles, and the RW metric. Note that this equation has the form of an energy equation, as if C is some sort of total energy. And as we shall see, C determines the fate of the universe, whether it has enough energy to expand forever, or it is bound and will collapse again. We like to write this energy equation in the form of Friedmann equation:

$$\dot{R}^2 = \frac{8\pi}{3} G\rho R^2 + C$$

Remember that the Hubble constant is $\dot{R}/R$ evaluated today. This means that today

$$H_0^2 = \frac{8\pi}{3} G\rho_0 + \frac{C}{R_0^2}$$

This sets the constant C.

Let us define a constant $\Omega_0 = 8\pi G\rho_0/3H_0^2$. Then we have

$$1 = \Omega_0 + \frac{C}{R_0^2 H_0^2}$$

$$C = R_0^2 H_0^2 (1 - \Omega_0)$$

Let's put this back in the equation for $\dot{R}$. We can write it as

$$\dot{R}^2 = \frac{8\pi}{3} G\rho_0 \frac{R_0^3}{R} + R_0^2 H_0^2 (1 - \Omega_0)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \Omega_0 H_0^2 \left(\frac{R_0}{R}\right)^3 + (1 - \Omega_0)\left(\frac{R_0}{R}\right)^2$$

$$\frac{H(z)}{H_0}^2 = \Omega_0 (1+z)^3 + (1 - \Omega_0)(1+z)^2$$

This is a simple formula for $H(z)$.

Note that in this equation, we are assuming that $\rho \sim (1+z)^{-3}$. This is true for matter, because the matter should be conserved.

5 Friedmann Equation

However, our toy model is missing an important aspect, which is the dynamical effect of the curvature of the universe. For this, we need to appeal to GR. We will not prove it, but GR yields the following two equations for the evolution of $R(t)$.

$$\ddot{R} = -\frac{4\pi G}{3} R \left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda R$$

$$\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

$$\kappa = 1/(R_c^2)$$

These look similar to the above, but with some new terms. First, the constant of integration in the Friedmann equation has been replaced by a term depending on the curvature κ. Second, we have terms that depend on pressure: matter: $p = 0$, and photon: $p = \rho c^2$. and on the cosmological constant Λ. Note what this Λ will do to your equation: (1) it is a repulsive force in the acceleration equation, while matter and pressure is always attracting, gravity slows things down. (2) by carefully choosing Λ and κ, you can have a stable solution, where $\ddot{R} = \dot{R} = 0$, at this stable radius $R = R_s$. This is called the Einstein-Lemaitre model, this is the reason why Einstein chose to use Λ in the original field equation, to obtain a stable solution, because it is very clear from these equations that otherwise the universe would have to expanse or contract, and have to decelerate.

Let's ignore p and Λ for now. The second equation gives us

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \kappa c^2/R^2 = \frac{8\pi G\rho_0}{3}(1+z)^3 - \kappa c^2(1+z)^2$$

We define $\Omega_0 = 8\pi G\rho_0/3H_0^2$ and find

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_0(1+z)^3 - \frac{\kappa c^2}{H_0^2}(1+z)^2$$

At the present-day ($z = 0$), the LHS is 1, so we must have

$$\Omega_0 - 1 = \frac{\kappa c^2}{H_0^2}$$

This says that the value of Ω_0 is directly related to the curvature of the universe. One does not have the freedom to pick an arbitrary geometry, density, *and* Hubble constant!

If $\Omega_0 = 1$, then $\kappa = 0$ and the geometry of the universe is flat. If $\Omega_0 < 1$, then $\kappa < 0$ and the geometry is hyperbolic (open). If $\Omega_0 > 1$, then $\kappa > 0$ and the geometry is spherical (closed).

6 Solution

Before we solve the Friedmann equation, let's look at its behavior. We can rewrite the Friedmann equation as:

$$\dot{R}^2 = H_0^2[\Omega_0(1/R - 1) + 1]$$

when $R \to \infty$,

$$\dot{R}^2 = H_0^2(1 - \Omega_0).$$

For $\Omega_0 = 1$ case, at $R = \infty$, $\dot{R} = 0$, so the universe will expand forever, but it eventually will come to a stop. This case is called the Einstein-de Sitter model, or the critical model.

For $\Omega_0 > 1$ or $\kappa > 0$ case, at some large R, $\dot{R}$ will go zero, so the universe will reach its max radius, and then begin to collapse. The expansion will never reach infinity and it will ends in the big crunch, this is the close model.

For $\Omega_0 < 1$ or $\kappa < 0$ case, $\dot{R}^2$ is always positive, the universe will expand forever, this is the open model.

Hence, in this simple formulation, we have direct connections between the density of the universe (as measured by Ω_0), the geometry of the universe, and the fate of the universe! However, these relations do breakdown when we introduce Λ. $\Omega_0 = 1$ is special. The required density $\rho_c = 3H_0^2/8\pi G$ is known as the critical density. $\Omega_0 = \rho_0/\rho_c$.

The ODE of Friedmann equation, when $\Lambda = 0$, can be solved analytically. $\Omega = 1$ case is easy, in this case:

$$\dot{R}^2 = H_0^2/R,$$

you can show easily that:

$$R = (t/t_0)^{2/3}$$

so the universe grow as a power law. And comparing these two equations, you can show easily:

$$t_0 = (2/3)H_0^{-1}.$$

Therefore, it reveals another physical meaning of Hubble constant: the inverse of H_0 gives the age of the universe.

Obviously, for $\Omega >$ and < 0, the universe is closed, or open, and grows slower and faster than the critical case, respectively. For open model, there is a simple case where the universe is completely empty, this is called the Milne model: $\Omega_0 = 0$, in this case, there is obviously no deceleration, and the universe will expand with a constant speed, $R(t) = H_0 t$.

Otherwise, the solution of Friedmann equation can be most conveniently written in parametric form: For $\Omega_0 > 1$:

$$R = a(1 - \cos\theta), t = b(\theta - \sin\theta),$$

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)}, b = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}.$$

Obviously, R will reach its max when $\theta = \pi$, and $t_{max} = \frac{\pi\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}$.

Similar solution can be found for the open model.

The relevance of the closed model solution is: as we will see later when we discuss galaxy formation, for a local area in the universe where the density is higher than the critical density, it can be treated like a closed universe, and it will collapse at this t_{max} timescale.

6.1 Λ model

The solution for Λ case is more complicated. But let's look at Friemann equation again, if $\Lambda \neq 0$,

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

There are three terms: the gravity term, curvature term and Λ term. See how different terms change with redshift. At $R >> 1$, the curvature term doesn't grow with R, and can be ignored; the gravity term goes slower than the Λ term, and can be ignored as well. So we will have that in the late stage of expansion in a Λ universe,

$$\dot{R} = \sqrt{\Lambda/3}R, R \propto \exp(\sqrt{\Lambda/3}t)$$

The universe is going to expand exponentially if $\Lambda > 0$! Note that this is true regardless of the curvature and as far as $\Lambda > 0$, even if it is relatively small, you can not escape the fate that the universe is going to expand exp, and it seems that we can't avoid this fate now. So very small (many billion years later), the universe is going to be exceedingly empty. We will meet this exp expansion again twice in this class, one about the dark energy, the other about inflation.

6.2 flatness problem

So how about when R is very small, in other words, at high redshift? For matter, we have $\rho \propto R^{-3} \propto (1+z)^3$, this term is going to dominate the Friedmann equation at early epoch, since the gravity term will go as $(1+z)$, curvature term is a constant, and Λ term goes as $R^2 \propto (1+z)^{-2}$. If we ignore the last two terms, then

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2,$$

and divide this by R^2, and note that $\rho_c = 3H^2/8\pi G$, we have $\Omega = \rho/\rho_c = 1$ at high redshift, regardless of what the current day curvature and cosmological constant is. There are two things to learn from this:

1. at high-redshift, the universe can always be approximated as being an Einstein-de Sitter model. Given the range of cosmological parameters we know today, the universe can be regarded as more or less being flat at $z > 4$ or so.

2. what is called the flatness problem. That is: if the curvature of the universe we measure today is close to zero, or Ω_0 is within a factor of 10 from unity, then at the very beginning, the universe must have been fine tuned to be very close to completely flat. Because if there is any very small deviation from flat at high redshift, it will be blow up more or less by a factor of $1+z$ now. The flatness of the universe is a strong function of z, so why we happen to be living at a time Ω_0 is not far from 1, while if the curvature is not zero, then Ω_0 can be equally possible to be 1000 or 1/1000? Come back to cosmological principle, this argues that if we believe that we are not living in a special time, then the universe must have always been precisely flat from the beginning to the end. This argument, or solution, for the flatness problem, is a strong motivation for inflation model, in fact, you can say that it is a prediction for such inflation model, because at that time, there was few evidence that the universe is flat. We will discuss this again later.

7 Energy Density

Now let's consider the relation between the $\ddot{R}$ and Friedmann equations. Differentiating the Friedmann equation gives

$$\ddot{R} = \frac{8\pi G}{3}\rho R + \frac{4\pi G}{3}R^2\frac{\dot{\rho}}{\dot{R}} + \frac{1}{3}\Lambda R$$

Subtracting the original GR acceleration gives

$$R\frac{\dot{\rho}}{\dot{R}} = -3\left(\rho + \frac{p}{c^2}\right)$$

We can write this as

$$R\frac{d\rho}{dR} = -3\left(\rho + \frac{p}{c^2}\right)$$

where we consider ρ as a function of R.

We are familiar with normal matter. Here, ρ is the familiar density and $p \ll \rho c^2$. So $\rho \propto R^{-3}$ as we have used.

What about other kinds of matter? Photons have a pressure that is a third of their energy density. We write $p = \rho c^2/3$. That means that $\rho \propto R^{-4}$. This corresponds to the number density of photons dropping as R^{-3} while their frequencies and hence energies decrease as R^{-1}.

In the case when $\rho \propto (1+z)^4$, it is easy to show that

$$\dot{R}^2 \propto 1/R^2$$

or

$$R \propto t^{1/2}$$

We often separate ρ into two pieces, the non-relativistic matter and the relativistic photons. We define $\Omega_M = \rho_m/\rho_c$ and $\Omega_r = \rho_r/\rho_c$ so that

$$\frac{\rho}{\rho_c} = \Omega_m(1+z)^3 + \Omega_r(1+z)^4$$

Note that since radiation goes as $(1+z)^4$, at some point it will dominate, so at very high redshift, it is always radiation dominated, and you can ignore mass when solving the dynamical evolution of the universe, while at later time, it is always matter dominated, as it is now. But in CMB era, it is radiation dominated. It is very important to remember that for matter: $\rho \sim (1+z)^{-3}$, and for radiation: $\rho \sim (1+z)^{-4}$.

8 Deceleration Parameter

Let's introduce one more parameter. We showed the Hubble constant is the expansion rate of the universe. So how do we define the acceleration, or more precisely, the deceleration, of the universe. We define:

$$q_0 = -\left(\frac{R\ddot{R}}{\dot{R}^2}\right)_{t_0}.$$

Substituting this definition to basic equations, when without Λ, and without pressure term, it is immediately obvious that

$$q_0 = \Omega_0/2.$$

So this shows that the deceleration of expansion is directly related to the mass density. Not to be surprised, the deceleration is caused by the gravity. But what it gives us is another way to measure the density of the universe, i.e., we know the expansion rate of the universe today; if we can somehow measure the expansion rate of the universe at high-redshift, then we can measure the deceleration and thus the density. This flavor of measuring Ω is called the geometrical measurement.

Note that in the absence of Λ, the universe always decelerate. But the above relation is not valid in case of Λ. As you probably know, the universe is actually accelerate.

9 Observations

We discussed the observables in cosmology, such as the different kind of distance, the co-moving, angular diameter, and luminosity distances, the volume, and look-back time. We derive their basic relation with RW metric, but since at that time we didn't know R(t), we can't write down their relation with redshift. Now we can. These relations are used for: (1) given a cosmology, and measured redshift, you can then derive the diameter, luminosity and time of a high-redshift source from the observed angular size, flux, and redshift.

(2) if you can get the diameter, luminosity etc. independently, then you can use these figures to constrain cosmology, like Hubble constant, density, cosmological constant etc.

So let's summarize some of the cosmological parameters we learned.

We can rewrite Friedmann equation:

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

by dividing $\dot{R}^2$ in both sides, and ask: $\Omega_Lambda = \Lambda/3H_0^2$, and $\Omega_k = -kc^2/H_0^2$, thus:

$$\left[\frac{H(z)}{H_0}\right]^2 = \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda.$$

This is the form of Friedmann equation I find most useful, because it connects Hubble constant with other cosmological parameters: density, curvature and cosmological constant. Obviously, at $z = 0$:

$$1 = \Omega_m + \Omega_k + \Omega_\Lambda,$$

What Friedmann equation gives us is the expansion history of the universe. Through it, we also introduced a number of cosmological parameters.

- $H_0 = \dot{R}/R|_0$, is the current expansion rate of the universe.
- t_0 is the current age of the universe since the BB.
- Ω_m is the current density parameter of the Universe.
- k is the curvature of the universe, deciding the geometry.
- Λ is the cosmological constant.
- q_0 is the deceleration parameter.

Clearly, they are not all independent. Indeed, they are tied by Friedmann equations. In matter dominated era, as the equation we showed above, the entire expansion history can be described by three parameters, including a scale (Hubble constant, or age of the universe), and two parameters that specify the relative contribution of matter (including dark matter), curvature and cosmological constant to the total energy-density budget of the universe. This is our **Robertson-Walker-Friedmann world model.** The most important task of a cosmologist is to understand what our world model is. And our biggest task is to test whether this world model, and which version of it, is supported by our observations. This is called the **cosmological tests.**

In fact, our cosmological test, or our cosmological model, includes even more parameters, because we are interested in not only the expansion history of the material dominated era, but (1) the state of the universe in radiation dominated era, i.e., CMB, (2) the growth of fluctuation in the universe.

WMAP 7-year summary. The main conclusion of this is that our cosmology can be described by essentially 7 parameters that fit all existing observations satisfactorily. These observations are (almost all):

(1) the expansion history of the universe, including measurements such as supernova, which we will discuss next lecture;

(2) the anisotropy of the CMB;

(3) the large scale structure in low-z universe, including measurements of galaxy clustering and structure of the IGM, as well as clusters of galaxies

And all of them can be fit by seven parameters (paper said six, because it is assuming a flat geometry: $H_0 = 70$, $\Omega_m = 0.28$, $\Omega_b = 0.05$, $k = 0$, and then two parameters, the zero point and slope that specify the density fluctuation power spectrum, and one parameter describes the cross section of free electrons to CMB that will affect CMB photons through Compton scattering. We will discuss Ω_b, and the other three later. For the moment, we will concentrate on the three parameters that appear in Friedmann equations, i.e., the expansion history of the universe, which we generally call the **classical cosmological tests**. We will discuss how to measure Ω_b in the BBN lecture, and discuss CMB in details, as well as large scale structure tests in details later.

In the next two lectures, we will talk about: (1) Hubble constant; (2) age; and (3) Ω_m very briefly, and (4) Λ and dark energy.