

Lagrange points for a point mass satisfy Equation 4.7 from S&G.

$$-\frac{GM}{(D-x)^2} \pm \frac{Gm}{x^2} - \Omega^2 \left(x - \frac{DM}{M+m} \right) = 0$$

With a "Dark Halo" potential, $\mathcal{M}(< r)$ is not constant and, as shown above, $\mathcal{M}(< r) \propto r$

An object at L_1, L_2 will have the same angular speed as the satellite (Ω^2 unchanged)

The major change will be in the first term, $-\frac{GM}{(D-x)^2}$ which represents the effect of \mathcal{M} at point x

$$\text{Therefore, } -\frac{GM(<D-x)}{|D-x|^2} \pm \frac{Gm}{x^2} - \Omega^2 \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = 0$$

$$\text{Since } \Omega^2 = \frac{G(\mathcal{M}(<D)+m)}{D^3} \text{ is unchanged, } -\frac{GM(<D-x)}{|D-x|^2} \pm \frac{Gm}{x^2} - \frac{G[\mathcal{M}(<D)+m]}{D^3} \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = 0$$

$$\frac{\mathcal{M}(<D-x)}{\mathcal{M}(<D)} = \frac{D-x}{D} \Rightarrow \mathcal{M}(<D-x) = \mathcal{M}(<D) \frac{D-x}{D}$$

$$-\frac{GM(<D)}{|D-x|^2} \frac{D-x}{D} \pm \frac{Gm}{x^2} - \frac{G[\mathcal{M}(<D)+m]}{D^3} \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = -\frac{GM(<D)}{D|D-x|} \pm \frac{Gm}{x^2} - \frac{G[\mathcal{M}(<D)+m]}{D^3} \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = 0$$

$$-\frac{GM(<D)}{D^2[1-\frac{x}{D}]} \pm \frac{Gm}{x^2} - \frac{G[\mathcal{M}(<D)+m]}{D^3} \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = 0$$

Using Taylor expansion ($m \ll \mathcal{M}(<D) \Rightarrow x \ll D$), $\frac{1}{1-\frac{x}{D}} \approx 1 + \frac{x}{D} + \frac{x^2}{D^2} + \dots$

$$\Rightarrow \approx -\frac{GM(<D)}{D^2} - \frac{GM(<D)x}{D^3} \pm \frac{Gm}{x^2} - \frac{G[\mathcal{M}(<D)+m]}{D^3} \left[x - \frac{DM(<D)}{\mathcal{M}(<D)+m} \right] = 0$$

$$-\frac{GM(<D)}{D^2} - \frac{GM(<D)x}{D^3} \pm \frac{Gm}{x^2} - \frac{GM(<D)x}{D^3} - \frac{Gmx}{D^3} + \frac{GM(<D)}{D^2} = -\frac{2GM(<D)x}{D^3} \pm \frac{Gm}{x^2} - \frac{Gmx}{D^3} = 0$$

$$\Rightarrow -2\mathcal{M}(<D)x^3 \pm mD^3 - mx^3 = 0$$

$$[2\mathcal{M}(<D) + m]x^3 = \pm mD^3$$

$$\Rightarrow x = \pm r_J = \pm D \left[\frac{m}{2\mathcal{M}(<D)+m} \right]^{1/3} \approx \pm D \left[\frac{m}{2\mathcal{M}(<D)} \right]^{1/3} \text{ [for } m \ll \mathcal{M}(<D)]$$