Problem 4 (S&G prob 3.20)

$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(c^2 - \frac{2GM_{BH}}{r}\right)\left(1 + \frac{L^2}{c^2r^2}\right) \equiv E^2 - 2\Phi_{eff}(r)$

a)

$\Phi_{eff}(r) = \frac{1}{2}\left(c^2 - \frac{2GM_{BH}}{r}\right)\left(1 + \frac{L^2}{c^2r^2}\right) = \frac{1}{2}\left(c^2 + \frac{L^2}{r^2} - \frac{2GM_{BH}}{r} - \frac{2GM_{BH}L^2}{c^2r^3}\right)$

For a Circular orbit, $\frac{d\Phi_{eff}(r)}{dr} = 0$

$\frac{d\Phi_{eff}(r)}{dr} = \frac{1}{2}\left(-\frac{2L^2}{r^3} + \frac{2GM_{BH}}{r^2} + \frac{6GM_{BH}L^2}{c^2r^4}\right) = -\frac{L^2}{r^3} + \frac{GM_{BH}}{r^2} + \frac{3GM_{BH}L^2}{c^2r^4}$

$= \frac{1}{r^4}\left[GM_{BH}r^2 + \frac{3GM_{BH}L^2}{c^2} - L^2 r\right] = \frac{1}{r^4}\left[GM_{BH}r^2 + L^2\left(\frac{3GM_{BH}}{c^2} - r\right)\right]$

$\frac{d\Phi_{eff}(r)}{dr} = 0 \Rightarrow GM_{BH}r^2 + L^2\left(\frac{3GM_{BH}}{c^2} - r\right) = 0 \Rightarrow r - \frac{3GM_{BH}}{c^2} = \frac{GM_{BH}r^2}{L^2}$

$GM_{BH}r^2$ is always positive and L^2 is always positive.

Therefore, for a Circular orbit,

$r - \frac{3GM_{BH}}{c^2} > 0 \Rightarrow r > \frac{3GM_{BH}}{c^2}$ and there are no Circular orbits with $r < \frac{3GM_{BH}}{c^2}$

b)

As shown above, for a Circular orbit,

$r - \frac{3GM_{BH}}{c^2} - \frac{GM_{BH}r^2}{L^2} = 0 \Rightarrow GM_{BH}c^2r^2 - L^2c^2r + 3GM_{BH}L^2 = 0$

$r = \frac{L^2c^2 \pm \sqrt{L^4c^4 - 4(GM_{BH}c^2 \times 3GM_{BH}L^2)}}{2GM_{BH}c^2} = \frac{L^2c^2 \pm \sqrt{L^4c^4 - 12G^2M_{BH}^2c^2L^2}}{2GM_{BH}c^2} = \frac{L^2c^2 \pm L^2c^2\sqrt{1 - \frac{12G^2M_{BH}^2}{c^2L^2}}}{2GM_{BH}c^2}$

$= \frac{L^2\left(1 \pm \sqrt{1 - \frac{12G^2M_{BH}^2}{c^2L^2}}\right)}{2GM_{BH}}$

For a real solution, $1 - \frac{12G^2M_{BH}^2}{c^2L^2} > 0$

$\Rightarrow c^2L^2 > 12G^2M_{BH}^2 \Rightarrow L > \sqrt{12}\frac{GM_{BH}}{c} \Rightarrow L > 2\sqrt{3}\frac{GM_{BH}}{c}$

The smallest possible radius with a circular orbit would be when $\sqrt{1 - \frac{12G^2M_{BH}^2}{c^2L^2}} = 0$

At this radius, $r = \frac{L^2}{2GM_{BH}}$

Since $L > \sqrt{12}\frac{GM_{BH}}{c} \Rightarrow r > \frac{\frac{12G^2M_{BH}^2}{c^2}}{2GM_{BH}} \Rightarrow r > \frac{6GM_{BH}}{c^2}$

For a Stable, Circular orbit, $\Phi_{eff}(r)$ must be at a minimum and, therefore, $\frac{d^2\Phi_{eff}(r)}{dr^2} > 0$

$\frac{d^2\Phi_{eff}(r)}{dr^2} = \frac{d}{dr}\left[\frac{GM_{BH}}{r^2} + \frac{3GM_{BH}L^2}{c^2r^4} - \frac{L^2}{r^3}\right] = \left[-2\frac{GM_{BH}}{r^3} - \frac{12GM_{BH}L^2}{c^2r^5} + 3\frac{L^2}{r^4}\right]$

$\frac{d^2\Phi_{eff}(r)}{dr^2} > 0 \Rightarrow \frac{1}{r^5}\left[-2GM_{BH}r^2 - \frac{12GM_{BH}L^2}{c^2} + 3L^2r\right] = \frac{1}{r^5}\left[3L^2(r - \frac{4GM_{BH}}{c^2}) - 2GM_{BH}r^2\right] > 0$

$\Rightarrow 3L^2(r - \frac{4GM_{BH}}{c^2}) - 2GM_{BH}r^2 > 0$

If $L = \sqrt{12}\frac{GM_{BH}}{c}$ *and* $r = \frac{6GM_{BH}}{c^2} \Rightarrow 36\frac{G^2M_{BH}^2}{c^2}(r - \frac{4GM_{BH}}{c^2}) - 2GM_{BH}r^2 =$

$36\frac{G^2M_{BH}^2}{c^2}(\frac{6GM_{BH}}{c^2} - \frac{4GM_{BH}}{c^2}) - 2GM_{BH}\left(\frac{6GM_{BH}}{c^2}\right)^2 = 36\frac{G^2M_{BH}^2}{c^2}(\frac{2GM_{BH}}{c^2}) - 2GM_{BH}\frac{36G^2M_{BH}^2}{c^4} = 0$

This is a borderline Stable, Circular orbit.

Therefore, Stable, Circular orbits must have $r > \frac{6GM_{BH}}{c^2}$, $L > 2\sqrt{3}\frac{GM_{BH}}{c}$