Midterm Solutions

Astronomy 400B

Problem 1 (20 points)

(a) Disk galaxies have an exponential surface density profile of stars: $\Sigma(R) = \Sigma_0 \exp(-R/b)$. Find out what is the total stellar mass of this galaxy. (5 points)

b) Find the rotation curve $v_c(R)$ for the exponential disk assuming that the gravitational force can be derived from a spherical treatment of the mass, i.e. $v_c^2 = GM(< R)/R$. Here we assume that the disk is self-gravitating, i.e., no dark matter, thus the rotation curve is a function of R and it is not flat (5 points).

(c) Find the radius at which the rotation curve peaks and give the maximum velocity. The numerical coefficients need not be exact; two significant figures is fine (5 points).

(a) **total mass**:

$$
\begin{aligned}
M &= \int_0^\infty 2\pi\Sigma_0 \exp(-R/b) R dR & (1)\\
&= 2\pi\Sigma_0 b^2 & (2)
\end{aligned}
$$

(b) **rotation curve.**

$$
\begin{aligned}
M(< R) &= 2\pi\Sigma_0 \int_0^R \exp(-R/b) R dR & (3)\\
&= 2\pi\Sigma_0 b^2 \int_0^{R/b} \exp(-x) x dx & (4)\\
&= 2\pi\Sigma_0 b^2 [e^{-x}(-x-1)]|_0^{R/b} & (5)\\
&= 2\pi\Sigma_0 b^2 [1-(1+R/b)\exp(-R/b)] & (6)\\
&= M_{total}[1-(1+R/b)\exp(-R/b)] & (7)
\end{aligned}
$$

so the rotation can be written as:

$$
v_c^2(R) = \frac{M_{total} G[1-(1+R/b)\exp(-R/b)]}{R}. \quad (8)
$$

(c) In order to find the maximum rotation speed, we find the place where $\partial v_c^2(R)/\partial R = 0$. It is convenient to define $x = (R/b)$. Then we have:

$$
v^2 \propto \frac{1-(1+x)\exp(-x}{x} \quad (9)
$$

$$\partial v_c^2(R)/\partial t \propto \frac{x[e^{-x} - e^{-x} + xe^{-x}] - [1-(1+x)e^{-x}]}{x^2} \quad (10)$$

$$\propto x^2 e^{-x} - 1 + (1+x)e^{-x} = 0 \quad (11)$$

We have:

$$(x^2 + x + 1)e^{-x} = 1 \quad (12)$$

this means: $e^x = x^2 + x + 1$, or

$$x = \ln(x^2 + x + 1) \quad (13)$$

We can solve this numerically, or we can solve it by iterating on: the equation above, with an initial guess of anything, say $x = 1$, after about three iteration, we find that $x \sim 1.8$. So the rotation speed reaches max at $R = 1.8b$. And the maximum rotation speed is: $V_{max}^2 = 0.53GM_{total}/b$.

Problem 2 (20 points)

(1) The Sagittarius dwarf galaxy is now about 15 kpc from the Galactic center. Assuming that the Galaxy has a flat rotation curve with $V(R) = 200$ km/s, find the mass of our Galaxy within that radius (5 points).

(2) The B band luminosity of Sagittarius is $10^7 L_\odot$. And the current radius is about 2 kpc. This dwarf galaxy is continuously being torn apart by the tidal interaction with the Galaxy. Show that this object is dark matter dominated. (10 points)

(4) People have also looked at NGC 5907, a bright edge-on galaxy (B = 11.1 mag) for such feature. The rotational speed of NGC 5907 is 160 km/s. Find the total luminosity of NGC 5907 in B band. What's the distance of NGC 5907? (5 points)

Answer:

(1) $M(< R) = Rv^2/G = 1.4 \times 10^{11} M_\odot$.

(2) tidal radius: $r_t = (m/3M)^{1/3}D$, so $m = 3M * (r_t/D)^3 = 10^9 M_\odot$, so the mass to light ratio is of the order 100.

(3) a. Tully-Fisher relation: $M_B = -7.5(\log_{10} V) - 1.97 = -18.5$. The total luminosity is then: $10^{-0.4*(-18.5-5.5)} = 4 \times 10^9 L_\odot$. The distance modulus is then: 11.7 - (-18.5) = 30.2. The distance is 11 Mpc.

Problem 3 (20 points)

(a) Prove that if a homogeneous sphere of a pressureless fluid with density ρ is released from rest, it will collapse to a point in time $t_{ff} = \frac{1}{4}\sqrt{3\pi/(2G\rho)}$. The time t_{ff} is the free-fall time of the system. (Hint: you might want to do the rest of the problem first if you think it might take you some time to prove this.) For a globular cluster with radius of 10 pc, and consist of 10^5 1 $M_\odot$ stars, what is the free-fall time? (10 points)

(b) For the same cluster, assume the velocity dispersion is 10 km/s, what is the crossing time? And what is the relaxation time? (Hint: Strong encounter radius is $r_s = 2Gm/V^2$). (5 points)

(c) Describe briefly how the two-body relaxation might change the structure of the cluster; what other effect you need to consider if the cluster is made of stars with different masses? (5 points)

(a) **free-fall timescale:** the simply way to think of it: the free-fall time is exactly 1/2 of the period of an ellptical orbit with $e = 1$, and the two foci being at $r = r_0$ and $r = 0$, so the semi-major axis is $r_0/2$. The you can find out the period using Kepler's third law:

$$T^2/a^3 = 4\pi^2/GM \tag{14}$$

So

$$T = 2\pi\sqrt{\frac{(r_0/2)^3}{GM}}. \tag{15}$$

The total mass $M = 4/3\pi\rho r_0^3$, plug this in:

$$T = 2\sqrt{\frac{3\pi}{32\rho G}} \tag{16}$$

Therefore, the free-fall time:

$$t_{ff} = T/2 = \sqrt{\frac{3\pi}{32\rho G}} \tag{17}$$

There is also a hard way. In this case, considering the conservation of total energy:

$$1/2(dr/dt)^2 + GM/r = GM/r_0 \tag{18}$$

so:

$$dt = \frac{dr}{\sqrt{2GM(1/r - 1/r_0)}} \tag{19}$$

so free-fall time is:

$$\begin{aligned} t_{ff} &= \int_{r_0}^{0} \frac{dr}{\sqrt{2GM(1/r - 1/r_0)}} && (20) \\ &= \frac{1}{\sqrt{2GM}} \int_{r_0}^{0} dr \sqrt{\frac{r_0 r}{r_0 - r}} && (21) \\ &= \sqrt{\frac{r_0}{2GM}} \int_{r_0}^{0} dr \sqrt{\frac{r}{r_0 - r}} && (22) \end{aligned}$$

Use the formula provided, the integral then is

$$[\sqrt{r}\sqrt{r_0 - r} - r_0 \arcsin\sqrt{(r_0 - r)/r_0})]|_{r_0}^{0} = 0 + r_0 \arcsin 1 = r_0\pi/2 \tag{23}$$

Plug this back in, you will get the same result.

The rest should be easy.

$$t_{ff} = 5.24 \times 10^{13}\text{sec} = 1.66 \times 10^6\text{yr}. \tag{24}$$

(2) **crossing time**

$$t_{crossing} = R/v = \frac{10 \times 3.09 \times 10^{18}\text{cm}}{10 \times 10^5\text{cm/s}} = 3.09 \times 10^{13}\text{sec} = 0.98 \times 10^6\text{yr} \tag{25}$$

Strong encounter radius is:

$$r_s = 2GM/V^2 = \frac{2 \times 6.67e-8 \times 1.99e33}{10^12} = 2.65e14\text{cm} \tag{26}$$

so

$$\ln\Lambda = \ln(R/r_s) = \ln(10 \times 3.09e18/2.65e14) = 11.7. \tag{27}$$

And the number density $n = \frac{3\times10^5}{4\pi10^3} = 23.9\text{pc}^{-3}$.

So the relaxation time is:

$$t_{\text{relax}} = \frac{2 \times 10^9}{11.7}\left(\frac{23.9}{10^3}\right)^{-1} = 7.1 \times 10^9\text{yr}. \tag{28}$$

(c) you need to say something about relaxation, evaporation and mass segregation.