AST 541 Lecture Notes: Classical Cosmology
Sep, 2018

In the next two weeks, we will cover the basic classic cosmology. The material is covered in Longair Chap 5 - 8. We will start with discussions on the first basic assumptions of cosmology, that our universe obeys **cosmological principles** and is expanding. We will introduce the **R-W metric**, which describes how to **measure** distance in cosmology, and from there discuss the meaning of measurements in cosmology, such as redshift, size, distance, look-back time, etc.

Then we will introduce the second basic assumption in cosmology, that the gravity in the universe is described by Einstein's GR. We will not discuss GR in any detail in this class. Instead, we will use a simple analogy to introduce the basic dynamical equation in cosmology, the **Friedmann equations**. We will look at the solutions of Friedmann equations, which will lead us to the definition of our basic cosmological parameters, the density parameters, Hubble constant, etc., and how are they related to each other.

Finally, we will spend sometime discussing the measurements of these **cosmological parameters**, how to arrive at our current so-called concordance cosmology, which is described as being geometrically flat, with low matter density and a dominant cosmological parameter term that gives the universe an acceleration in its expansion. These next few lectures are the foundations of our class. You might have learned some of it in your undergraduate astronomy class; now we are dealing with it in more detail and more rigorously.

1 Cosmological Principles

The crucial principles guiding cosmology theory are homogeneity and expansion.

The cosmological principle says: We are not located at any special location in the Universe. The other way to put it is that the universe is homogeneous and isotropic.

Isotropy. This is not so obvious in the nearby galaxies, but distant galaxies, radio sources, and the CMB all show isotropy.

We could be living at the center of a spherically symmetric universe with a radial profile. However, we probably are not at a special point. If two observers both think that the universe is isotropic, then it is in fact homogeneous.

Homogeneity is a stronger result, because it means that we can apply our physical laws from

Earth. Homogeneity is harder to prove. But if we have a 3-D map of the sky, as in SDSS, we can certainly ask the question of how uniform (not isotropic) the universe is on a fairly large scale, which proves homogeneity. In addition, we see high-z object spectra, which follow the same quantum physics law; motions in distant clusters of galaxies follow the same gravity laws. Also, using quasar absorption lines, people can measure the CMB temperature at high-z, as the excitation temperature in low-T environment, which follows (1+z) etc

Both of these ideas only apply on large scales, hundreds of Mpc. As we will see, the universe is several Gpc across, so there are many patches in the universe with which to define homogeneity.

For the next while, we will consider the universe to be straightly isotropic and homogenous to study the basic principles of cosmology. We will then try to relax this, since, as you know, if the universe is really 100% homogenous, then there will be no gravitational fluctuation and no grow of fluctuation, and therefore no formation of galaxies etc. So in later part, we will consider how the universe is different from completely homogenous, and how these fluctuations grow with time, from the very small fluctuations in CMB to forming large galaxies and large scale structure of galaxies we see today.

Before we move on, for completeness, we should discuss a stronger version of cosmological principles, i.e., the so-called **perfect CP**, of Bondi and Gold, and Hoyle in 1940s. This requires the invariance not only under rotation and displacements in space, but also as a function of time. In other words, the universe looks the same at all directions, all positions and all times. This hypothesis led to the **steady state cosmology**, which at least until the discovery of CMB in 1965, was a valid, competing, and some might say, sometimes dominating cosmology. However, people did know at that time that the universe was expanding. To solve that, steady state cosmology requires that the universe is in a continuous creation of matter to keep the mean density of the universe constant. You might think that this is odd; but so is dark energy or dark matter, so I don't think people was scared about this unknown physics. Even after the discovery of CMB, steady state cosmology didn't die easily and its proponents invented ways to explain CMB. It was really only after COBE etc., with the smoothness of the CMB, which is very difficult to explain without an extremely optically thick phase of the universe at high-z, that steady state cosmology finally left any main stream literature, and we should not discuss it again in this class.

2 Metric

The next key concept of modern cosmology is **metric, which specifies the distance between two points, in space or in 4-d space time**. Our goals here are: (1) how to express distance as a function of difference in coordinates; (2) how to incorporate time in our expressions and (3) how to incorporate expansion in the expressions.

In 1935, Robertson and Walker were the first to derive (independently) the form of the metric of space-time for all isotropic homogeneous, uniformly expanding model of the Universe. The form of the metric is independent of the assumption that the large-scale dynamics of the universe are described by GR. Whatever form of the physics of expansion, the space-time metric must be of RW form, because, as we will see now, the only assumptions are isotropy and homogeneity. Now let's see how to derive it.

Distant galaxies are observed to be redshifted, as if they are moving away from us.

The recessional velocity is proportional to the distance of the galaxy. This is known as the Hubble law.

This redshifting is easy to understand in a homogeneous universe. All observers would agree on the expansion and the Hubble constant.

However, the fact that the universe used to be smaller does cause some computational complications in the coordinate system.

We won't be deriving cosmology from general relativity in this course, but we do need to use these coordinate systems.

To translate our cosmological ideas into math and physics, we need to define coordinate systems.

Key concept is the metric, which specifies the distance between two points (in space or space-time).

Imagine a 2-dim flat space. We are used to measuring the distance between 2 points by the Pythagorean theorem.

$$(\Delta\ell)^2 = (\Delta x)^2 + (\Delta y)^2$$

If the two points are infinitesimally close together, then we can write the distance as

$$d\ell^2 = dx^2 + dy^2$$

However, the idea of differencing the coordinates only works in Cartesian coordinates. If we switch to polar coordinates, then the distance between two neighboring points is

$$d\ell^2 = dr^2 + r^2 d\theta^2$$

The formula for measuring the distance is called the metric. In general, there could be mixed terms, e.g. $dr\,d\theta$, but we won't need these much. Sometimes, one writes this as a matrix $d\ell^2 = g_{\mu\nu}dx^\mu dx^\nu$, in which case the matrix $g_{\mu\nu}$ is called the metric tensor. Note that the metric can depend on the coordinates themselves.

Changing between coordinates systems can be done by differentiating the relations. In this case, $x = r\cos\theta$ and $y = r\sin\theta$ give $dx = dr\cos\theta - r\sin\theta\,d\theta$, etc.

To compute the length of a curve in our space, we integrate the length $d\ell$ along the curve.

Next, imagine the surface of a sphere of radius R_c. We are interested in homogenous space, following the cosmological principles. The plane is an obvious example of a space in which all points are the same. The sphere is another example.

Here, using spherical coords, we have

$$d\ell^2 = R_c^2 d\theta^2 + R_c^2 \sin^2\theta\, d\phi^2$$

We are interested in homogenous space, following the cosmological principles. The plane is an obvious example of a space in which all points are the same. The sphere is another example.

This differs from our result with polar coordinates on the flat space. One of the important results from differential geometry is that no change of coordinates can turn one metric into the other. In other words, there is no coordinate system on the sphere that looks like the Cartesian metric. This is because the sphere has a curvature that cannot be done away with by a coordinate transformation. Indeed, this leads one to the result that the metric gives a space its curvature without any reference to the embedding space of the manifold. Specifying the distances between all neighboring points defines the curvature of a space.

Just because one doesn't need an embedding space doesn't mean that they aren't useful. For example, proving that the sphere is a homogeneous space (the curvature is the same around any point) is obvious if one pictures the sphere in 3-space, but looks pretty daunting if one just looks at the metric. Differential geometry tells one how to define curvature, however, and this would make the metric proof possible.

However, despite the apparent complexity, near any non-singular point, one can always define a coordinate system that appears Cartesian. In our case, near the North Pole, if $r \ll R_c$ we will recover the polar coordinate metric, which we know is just a transformation of the Cartesian one.

Our spherical metric yields circles of constant r have smaller circumferences than flat space would imply, $2\pi R_c \sin(r/R_c)$ instead of $2\pi r$.

The opposite case, one in which circles are larger than expected, is also interesting. We write

$$d\ell^2 = dr^2 + R_c^2 \sinh^2(r/R_c)\, d\phi^2$$

Such a metric is known as a hyperbolic geometry. Here, we are using the hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

instead of

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

.

This is similar to the sphere in that one can translate within the manifold and recover the same metric. Unfortunately, this manifold does not have a simple shape in 3-dimensions. The extra circumference makes one imagine a plane crinkled like a saddle, however this is only a local approximation. The key point is that the directions of curvature are in opposite directions. Unlike the sphere, r extends to infinity and the space have infinite area. We will concentrate on the spherical case, the hyperbolic case is completely symmetric to the spherical case, just change sin to sinh.

Now, let's consider 3-dimensional spaces. The flat space metric is easy

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

In spherical coordinates, we have

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Again, the 3-sphere of radius R_c in 4 dimensions has full homogeneity. The metric is

$$d\ell^2 = R_c^2(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2)$$

If we choose $r = R_c\chi$, we reach

$$d\ell^2 = dr^2 + R_c^2 \sin^2(r/R_c)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

The term in brackets looks like the standard angular distance in spherical coords. Again, as before, if $r \ll R_c$, we recover the flat space metric, whereas at larger r the circumference is smaller than one expects. We will concentrate on the spherical case, the hyperbolic case is completely symmetric to the spherical case, just change sin to sinh.

These are the only homogeneous 3-dimensional spaces! Of course, one can always change coordinates to disguise the familiar form, but any homogeneous space can be brought to one of these three forms.

There is a common manipulation of these forms, in which the curvature anomaly is put in the radius rather than in the circumference. Write $x = R_c \sin(r/R_c)$ (sinh in the hyperbolic case). Then

$$dx = \cos(r/R_c)dr = \sqrt{1-\sin^2(r/R_c)}dr = \sqrt{1-x^2/R_c^2}dr$$

We define $\kappa = 1/R_c^2$ as the square of the radius curvature. The metric becomes

$$d\ell^2 = \frac{dx^2}{1-\kappa x^2} + x^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

The hyperbolic case looks the same, but with $\kappa = -1/R_c^2 < 0$. Now all three homogeneous spaces can be written in the same form with different signs of κ (0 for the flat case).

We have now generated some coordinate systems for homogeneous spaces. Because we want to discuss time evolution, we must get time into the picture!

In special relativity, we made use of a new length in space-time.

$$ds^2 = dt^2 - \frac{d\ell^2}{c^2}$$

We thought of the motions of particles as curves in space-time in which the time direction was treated specially in the metric. Recall that the distance ds is invariant under Lorentz transformations.

Photons move as $ds = 0$. Massive particles have $ds > 0$.

We want to include the expansion of the universe. For this, we will choose coordinates that move with the particles. In other words, a given particle that is at rest with respect to the expansion will have constant coordinates. And the expansion of the universe is reflected in the change of the size/scale of the universe as a whole. This coordinate system is called a comoving system, In this system, each galaxy is labelled by r, which is called the comoving radial distance coordinate. It is fixed, i.e., not changing with time. Note that we are dealing with isotropic and homogeneous universe here, so there is no peculiar velocity. The expansion of the universe is defined in a quantity $R(t)$, the scale factor.

In these coordinates, the spatial metric is

$$d\ell^2 = R(t)^2\left[dr^2 + R_c^2\sin^2(r/R_c)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

where $R(t)$ will simply change all the distances by some function of time. Note that R_c is time-independent (Longair calls this $\mathcal{R}$). It is conventional to pick $R(t) = 1$ at the present day, so the coordinates reflect the present-day scale of the universe. This is called the comoving frame.

The full space-time metric is then

$$ds^2 = dt^2 - \frac{R(t)^2}{c^2}\left[dr^2 + R_c^2 \sin^2(r/R_c)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

This is known as the Robertson-Walker metric. Remember that sin goes to sinh in the hyperbolic case. And note that when we ask R_c, the radius of the curvature goes to infinity, then everything goes back to the flat case.

We will concentrate on the spherical case, the hyperbolic case is completely symmetric to the spherical case, just change sin to sinh.

It is also common for $R(t)$ to be written $a(t)$. $R(t)$ is called the expansion factor. This is a very important formula! From this, one calculates all distances and volumes in cosmology. Note that while can compute between any two points, it is often simplest to put one of the points at the origin $r = 0$. This avoids the problem of solving for the equation of straight line.

3 Measurements in Cosmology

We have not yet specified the expansion factor $R(t)$. Computing it requires specifying all the gravitating mass in the universe. For now, we leave it general because one can compute a number of important quantities for arbitrary $R(t)$.

3.1 Redshift

First, consider the propagation of photons. For photons, $ds = 0$. We will place ourselves at the origin $r = 0$ and consider the arrival of photons from a source at location r_1. Obviously, the photon travels purely radially in our coordinate system, so $d\theta = d\phi = 0$. We have

$$\frac{c}{R(t)}dt = -dr$$

where the minus sign is because the photon is traveling toward the origin. Here

$$r = \int C/R(t)dt$$

is the comoving distance.

Let's imagine two signals sent to us. One leaves at t_1 and is received at t_0; the next is sent at $t_1 + \Delta t_1$ and is received at $t_0 + \Delta t_0$.

Let's integrate along the path to find the travel time.

$$\int_{t_1}^{t_0} \frac{c}{R(t)}dt = -\int_r^0 dr = r$$

The second signal is

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

Note that the coordinate distance traveled is the same; we have put the expansion of the universe in the $R(t)$ term, while the source and observer sit at fixed r.

Setting these equal gives

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c}{R(t)} dt = \int_{t_1}^{t_0} \frac{c}{R(t)} dt + \frac{c\Delta t_0}{R(t_0)} - \frac{c\Delta t_1}{R(t_1)}$$

So

$$\Delta t_0 = \Delta t_1 \frac{R(t_0)}{R(t_1)}$$

The difference of the arrival times has been dilated relative to the difference of the departure times by a factor that is the ratio of the expansion factors at the two times. You can understand it as time dilation in special relativity, but the way we derive it, this has nothing to do with relativity. If we imagine that our departure times were successive crests of an electromagnetic wave, then the period of the light must be altered by a factor $R(t_0)/R(t_1)$.

The light appears Doppler shifted. Its wavelength is changed by a factor $R(t_0)/R(t_1)$. We normally define the redshift by $\lambda_{obs} = \lambda_{emit}(1+z)$. This means that

$$1 + z = R(t_0)/R(t_1)$$

Again, the expansion of the universe causes this redshift without reference to gravity. One can think of the wavelength of the light being stretched by the expansion. Light from redshift $z = 1$ was emitted when the universe was half its present size.

If we consider the redshift of light reaching us, then our convention $R(t_0) = 1$ means that $1 + z = 1/R(t_1)$. This means that redshift can be thought of as a time variable $z(t)$.

Locally, we like to think of the redshift as a velocity effect. The Hubble law is written as a ratio of velocity to distance.

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

If $t_0 - t_1$ is small compared to the time scales over $R(t)$ changes, then we have $c(t_0 - t_1) = R(t_0)r$, which makes sense!

Now for the velocities.

$$z = \frac{R(t_0)}{R(t_1)} - 1 \approx \frac{R(t_0)}{R(t_0) + (t_1 - t_0)\dot{R}(t_0)} - 1 = \frac{1}{1 - (t_0 - t_1)\dot{R}(t_0)/R(t_0)} - 1 \approx (t_0 - t_1)\frac{\dot{R}(t_0)}{R(t_0)}$$

For small redshifts, the velocity is $v = cz$, so the ratio of the velocity to the distance $R(t_0)r$ is $H_0 = \dot{R}(t_0)/R(t_0)$.

One can define the Hubble constant as a function of time $H(t) = \dot{R}/R$. This relates the observed expansion to the behavior of $R(t)$.

Cosmological redshifts take on another form when one considers particle propagation. It turns out that we can think of the shift in wavelength as a shift in the momentum of the particles *relative to the comoving frame*. In an expanding universe, the momenta of particles drops as $1/a(t)$. This is a fully relativistic statement: it affects massless and massive particles alike. Since photons have energy proportional to momentum, one can see that the momentum redshifting statement is equivalent to the more familiar redshifting of the wavelength.

For non-relativistic particles, the drop in momentum means that the velocity is decreasing. What does this mean? As the particle moves, it "catches" up with neighboring particles in the expanding universe. Relative to these particles, it is moving slower.

This view is particularly useful when considering the transition of a particle from relativistic ($v \approx c$) to non-relativistic.

3.2 Distance

What distances do we actually measure in cosmology? We have seen one, the **comoving distance** traveled in a given time

$$\int_{t_1}^{t_0} \frac{c}{R(t)} dt = -\int_r^0 dr = r$$

Note that if we change integration variables from t to $z = 1/R(t) - 1$, then we have $dz = -(dR/dt)/R^2\, dt = -H(t)(1+z)\, dt$. So

$$r = \int_0^z \frac{c\, dz}{H(t)}$$

We will eventually have simple formulae for $H(z)$, but it's arbitrary for now.

However, we will find that this distance isn't necessarily the one that we measure! In fact, we can't measure r, the comoving distance at all.

How do we measure distance in cosmology? We can't go there and have a ruler. We more or less use a combination of two ways. First, if we know the true angular diameter of something by other means, and we can measure the angular diameter observed using telescope, then we get the distance. Examples are like the definition of parallex, and the moving cluster method. This is called **the angular diameter distance**, $D_A = d/\Delta\theta$.

Second, if we know the true luminosity of something by other means, and we can measure the apparent luminosity, then we get the distance. Examples are like Cepheids, T-F etc. This is called **the Luminosity Distance**. $f = L/(4\pi D_L^2)$.

Now let's see how to measure these distances in a cosmological context.

Consider an object of some known physical diameter at some redshift. What angle on the sky do we measure?

We will imagine ourselves at $r = 0$ and consider the object to be extended in the θ direction. In this case, our object is lying tangentially, so $dr = d\phi = 0$. Its diameter is then

$$d = R(t)R_c \sin(r/R_c)\Delta\theta$$

We would expect that $d/\Delta\theta$ would be the distance to the source. We define this ratio as the "angular diameter distance" D_A.

$$D_A = R(t)R_c \sin(r/R_c) = \frac{R_c \sin(r/R_c)}{1+z}$$

We computed $r(z)$ above.

In the flat cosmology, $D_A = r/(1+z)$. Note that the expansion causes the transverse distances to differ from the radial ones!

Next, let's consider a source of some luminosity L at some redshift z. We would expect the flux to be $L/4\pi D^2$. What do we actually get?

Place the source at the origin. How much solid angle does our telescope subtend? Now the question reversed, it is the angle that the distant galaxy will measure now. The metric says the transverse distance today corresponding to a given angle is $R(t_0)R_c \sin(r/R_c)\Delta\theta$. So the solid angle of an area A is $A[R_c \sin(r/R_c)]^{-2}$. The flux we receive per unit area is then $f = L/(4\pi[R_c \sin(r/R_c)]^2)$.

Is that all? There are other effects, however. The source is emitting a given number of photons per second, but we receive these photons in a time that is stretched by $R(t_0)/R(t_1)$. So the number flux is reduced by $1 + z$. Moreover, each photon is redshifted and has its frequency reduced by $1 + z$. So the energy flux is reduced by $(1 + z)^2$.

We receive a flux

$$f = \frac{L}{4\pi[R_c \sin(r/R_c)]^2(1+z)^2}$$

If we define the "luminosity distance" by $f = L/4\pi D_L^2$, then

$$D_L = R_c \sin(r/R_c)(1+z) = D_A(1+z)^2$$

Warning: if one is concerned with the flux per unit frequency, then $f_\nu \neq L_{\nu(1+z)}/4\pi D_L^2$. The flux over some range in frequency does scale as D_L^{-2}, but the range of frequency itself is scaling as $(1+z)^{-1}$. So $f_\nu = L_{\nu(1+z)}(1+z)/4\pi D_L^2$.

If the flux received goes as $f \sim 1/D_L^2$ while the angle subtended goes as $\Omega \sim 1/D_A^2$, then the surface brightness is going as $SB \sim f/\Omega \sim D_A^2/D_L^2$. This always goes as $(1+z)^{-4}$ without any reference to the density of the universe. High-redshift objects have lower surface brightness. This has important consequence in high-redshift galaxy observations!

There are a couple of other distances that can be measured, but they don't occur very often. The luminosity distance and angular diameter distance are the total major ways that we express distance in cosmology. Note that because the expansion of the universe, and the fact that how we define distance depends on the expansion in different ways, there are many different distances in cosmology, and for anything with substantial redshift, say $z > 0.1$, they are different. So when you say something is N Mpc away, it is not meaningful unless you see what it means. Note that all these distance measurements also involve calculating the comoving distance $r = \int cdt/R(t)$, and R_c, the radius of curvature of the universe, which we don't know unless we (1) find out the curvature, and (2) solve the evolution of $R(t)$. A much better way is to write down things in redshift, and if you know the evolution of $R(t)$, then we can calculate distance in any way you want. However, the true importance of talking about these distances is that given redshift, it gives you the relation between size and angular diameter, between apparent and absolute flux. That's what you need when you are measuring a distant object and try to interpret the data!

3.3 Volume

How much comoving volume is there in a given solid angle and redshift range? Here we are asking comoving volume, so we can forget about the scale factor $R(t)$ for the moment.

$$dV_{com} = R_c^2 \sin^2(r/R_c) d\Omega dr$$

from the metric. Why we need the volume? Because we want to get the spatial density of, say, galaxies. So imagine that we carry out a survey, and we count the galaxies within a redshift interval dz, and a solid angle $d\Omega$, we want to know how to get the density of these galaxies, so we need to understand how $dz \times d\Omega$, which is the thing that we measure. It is related to dV, which is the thing that we need for calculating volume.

Differentiating

$$r(z) = \int_0^z \frac{c\, dz}{H(t)}$$

gives $dr/dz = c/H(t)$.

$$\frac{dV_{com}}{d\Omega dz} = \frac{R_c^2 \sin^2(r/R_c) c}{H(t)}$$

3.4 Look-Back Time

Finally, we would like to calculate the look-back time to a given redshift, the time from a certain redshift to the current epoch, which is something we are clearly interested in, for example, to calculate how much time there is for the stars in a galaxy to evolve if it was formed at $z = 5$.

From $dt = -dr\ R(t)/c$, we have

$$t(z) = \int_0^r \frac{dr}{c} R(t) = \int_0^z \frac{dz}{H(t)} \frac{1}{1+z}$$

3.5 Horizons

Horizons — returning to the propagation of light in our metric, it is possible that the integral

$$r = \int_0^z \frac{c\, dz}{H(t)}$$

converges to a finite answer as $z \to \infty$. This is the beginning. This means that only a finite volume of the universe is within our past light cone. In other words, there are spatial points from which we cannot have received light and hence cannot be affected by causal physics. This effect is called a **particle horizon**, it is asking the question: since the universe has a beginning, and it takes time for light to travel, which is the fastest way to establish causal communications, what is the maximum distance over which causal communication could have taken place at a certain epoch. In other words, what is the distance a light signal could have travelled from the origin of the big bang (if there were one) at t=0, by the epoch t. In a few lectures, when we are discussing the generation of CMB, we will encounter a rather severe problem for cosmologist, namely, the particle horizon size at the time when CMB was generated (z=1100) is only 2 deg or so on the sky, on the other hand, CMB is very uniform across the sky. So this question gave rises to the idea of inflation, is, then, how one side of the universe were able to coordinate with the other side of the universe and had the same T, if there were no chance for any causal communication.

There is another possible horizon in which we compute the radius a light ray can reach in the infinite future. If

$$\int_{t_0}^{\infty} \frac{c\,dt}{R(t)}$$

is finite, then news of events beyond that radius will never reach us. This is called an event horizon. In black holes, the event horizon is the surface into which we cannot gain information from inside. In this case, the event horizon is an enormous sphere, and we cannot get information from outside of it! In particular, in a Λ-dominated universe, as we will find out, the universe accelerate exponentially in later times, and this integral can be finite, which means that we will eventually be in some kind of isolated bubbles. It also has important consequence in cosmology.

We have now studied the Robertson-Walker metric for a general $R(t)$. The behavior of $R(t)$ depends on the gravitational attractions of the homogeneous matter. In the next two lectures, we will introduce the basic equation for the dynamical expansion of the universe, the Friedmann equation. It is the expansion that obeys GR.

4 Friedmann Equations

4.1 Newtonian Analogue

Usually, one derives this from General Relativity, but let us first consider a Newtonian toy problem. Consider a homogeneous density distribution in uniform expansion. If we pick a origin and draw a sphere around it, then we might calculate the gravitational force by Gauss's law. This means that the material outside the sphere doesn't contribute. In detail, we haven't proven this (nor can we).

We will let the radius of the sphere move with its boundary particles. The radius of the sphere is denoted $R(t)$. The mass inside it is constant. If the density today is ρ_0 and the radius today is R_0, then the mass is $4\pi\rho_0 R_0^3/3$. The acceleration is then

$$\ddot{R} = -\frac{4\pi}{3} G\rho_0 \frac{R_0^3}{R^2}$$

As the sphere expands, the density must scale as $\rho = \rho_0 R_0^3 R^{-3}$. So we also have

$$\ddot{R} = -\frac{M}{R^2} = -\frac{4\pi}{3} G\rho R$$

This is a second-order ODE for the expansion of our toy universe. We must specify two boundary conditions, which we will take as the radius and Hubble constant today.

$$R|_{t_0} = R_0$$

$$(\dot{R}/R)|_{t_0} = H_0$$

Using a familiar DE trick:

$$\ddot{R} = d\dot{R}/dt = d\dot{R}/dR\dot{R}$$

We can integrate this equation and get:

$$\frac{1}{2}\dot{R}^2 = \frac{4\pi}{3}G\rho_0\frac{R_0^3}{R} + C/2$$

Here C is a constant of integral. This ODE is the famous Friedmann equation, which provides the evolution of the scale factor. It is the dynamical equation of the evolution of the universe. Think of it as the Newton's law in cosmology. In fact, we just derived from Newton's law in a homogeneous sphere case. It can be derived from GR, Einstein's field equation, assuming cosmological principles, and the RW metric. Note that this equation has the form of an energy equation, as if C is some sort of total energy. And as we shall see, C determines the fate of the universe, whether it has enough energy to expand forever, or it is bound and will collapse again. We like to write this energy equation in the form of Friedmann equation:

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 + C$$

Remember that the Hubble constant is $\dot{R}/R$ evaluated today. This means that today

$$H_0^2 = \frac{8\pi}{3}G\rho_0 + \frac{C}{R_0^2}$$

This sets the constant C.

Let us define a constant $\Omega_0 = 8\pi G\rho_0/3H_0^2$. Then we have

$$1 = \Omega_0 + \frac{C}{R_0^2 H_0^2}$$

$$C = R_0^2 H_0^2 (1 - \Omega_0)$$

Let's put this back in the equation for $\dot{R}$. We can write it as

$$\dot{R}^2 = \frac{8\pi}{3}G\rho_0\frac{R_0^3}{R} + R_0^2 H_0^2(1 - \Omega_0)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \Omega_0 H_0^2 \left(\frac{R_0}{R}\right)^3 + (1 - \Omega_0)\left(\frac{R_0}{R}\right)^2$$

$$\frac{H(z)}{H_0}^2 = \Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2$$

This is a simple formula for $H(z)$.

Note that in this equation, we are assuming that $\rho \sim (1+z)^{-3}$. This is true for matter, because the matter should be conserved.

Recall that we were able to express our distance-redshift relations in terms of integrals over $H(z)$. For example, we had the coordinate distance

$$r = \int_0^z \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^z \frac{dz}{[\Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2]^{1/2}}$$

4.2 Friedmann Equations in GR

However, our toy model is missing an important aspect, which is the dynamical effect of the curvature of the universe. For this, we need to appeal to GR. GR expresses gravitational forces as curvatures in space-time, so that particles that are not being acted on by non-gravitational forces fall along straight lines in space-time that appear as accelerating trajectories. Recall that the curvature of a manifold is completely specified by the (spatially-dependent) metric. The primary equation of GR equates a complicated function of that metric to the distribution of matter and momentum:

$$f(g_{\mu\nu}) = 8\pi G T_{\mu\nu}$$

The function on the left-hand side constructs a 2-tensor from the second derivatives of the metric. The right-hand side contains the stress-energy tensor. This tensor contains the spatial distribution and flux of energy and momentum. It is a symmetric tensor, implying that the energy flux is equal to the momentum density.

For an ideal gas in its rest-frame coordinate system, the stress-energy tensor is diagonal, with the energy density in the time-time position and the pressure in the three diagonal space-space positions. We will use this ideal gas form, but note that it is an approximation.

We will not prove it, but GR yields the following two equations for the evolution of $R(t)$.

$$\ddot{R} = -\frac{4\pi G}{3} R\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda R$$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

$$\kappa = 1/(R_c^2)$$

These look similar to the above, but with some new terms. First, the constant of integration in the Friedmann equation has been replaced by a term depending on the curvature κ. Second, we have terms that depend on pressure: matter: $p = 0$, and photon: $p = 1/3\rho c^2$, and on the cosmological constant Λ.

This Λ is something that you might remember if you still remember your Einstein's field equation

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G/c^4 T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, and R is the curvature scalar. And recall that Einstein's the original purpose was to introduce a term to stabilize the field equation to avoid the expansion solution.

Note what this Λ will do to your equation: (1) it is a replusive force in the acceleration equation, while matter and pressure is always attracting, gravity slows things down. In other words, you need Λ to balance things. (2) by carefully choosing Λ and κ, you can have a stable solution, where $\ddot{R} = \dot{R} = 0$, at this stable radius $R = R_s$. This is called the Einstein-Lemaitre model, this is the reason why Einstein chose to use Λ in the original field equation, to obtain a stable solution, because it is very clear from these equations that otherwise the universe would have to expanse or contract, and have to decelerate.

Let's ignore p and Λ for now. The second equation gives us

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \kappa c^2/R^2 = \frac{8\pi G\rho_0}{3}(1+z)^3 - \kappa c^2(1+z)^2$$

We define $\Omega_0 = 8\pi G\rho_0/3H_0^2$ and find

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_0(1+z)^3 - \frac{\kappa c^2}{H_0^2}(1+z)^2$$

At the present-day ($z = 0$), the LHS is 1, so we must have

$$\Omega_0 - 1 = \frac{\kappa c^2}{H_0^2}$$

This says that the value of Ω_0 is directly related to the curvature of the universe. One does not have the freedom to pick an arbitrary geometry, density, *and* Hubble constant!

If $\Omega_0 = 1$, then $\kappa = 0$ and the geometry of the universe is flat. If $\Omega_0 < 1$, then $\kappa < 0$ and the geometry is hyperbolic (open). If $\Omega_0 > 1$, then $\kappa > 0$ and the geometry is spherical (closed).

4.3 Solution

Before we solve the Friedmann equation, let's look at its behavior. We can rewrite the Friedmann equation as:

$$\dot{R}^2 = H_0^2[\Omega_0(1/R - 1) + 1]$$

when $R \to \infty$,

$$\dot{R}^2 = H_0^2(1 - \Omega_0).$$

For $\Omega_0 = 1$ case, at $R = \infty$, $\dot{R} = 0$, so the universe will expand forever, but it eventually will come to a stop. This case is called the Einstein-de Sitter model, or the critical model.

For $\Omega_0 > 1$ or $\kappa > 0$ case, at some large R, $\dot{R}$ will go zero, so the universe will reach its max radius, and then begin to collapse. The expansion will never reach infinity and it will ends in the big crunch, this is the close model.

For $\Omega_0 < 1$ or $\kappa < 0$ case, $\dot{R}^2$ is always positive, the universe will expand forever, this is the open model.

Hence, in this simple formulation, we have direct connections between the density of the universe (as measured by Ω_0), the geometry of the universe, and the fate of the universe! However, these relations do breakdown when we introduce Λ. $\Omega_0 = 1$ is special. The required density $\rho_c = 3H_0^2/8\pi G$ is known as the critical density. $\Omega_0 = \rho_0/\rho_c$.

The ODE of Friedmann equation, when $\Lambda = 0$, can be solved analytically. $\Omega = 1$ case is easy, in this case:

$$\dot{R}^2 = H_0^2/R,$$

you can show easily that:

$$R = (t/t_0)^{2/3}$$

so the universe grow as a power law. And comparing these two equations, you can show easily:

$$t_0 = (2/3)H_0^{-1}.$$

Therefore, it reveals another physical meaning of Hubble constant: the inverse of H_0 gives the age of the universe.

Obviously, for $\Omega >$ and < 0, the universe is closed, or open, and grows slower and faster than the critical case, respectively. For open model, there is a simple case where the universe is completely empty, this is called the Milne model: $\Omega_0 = 0$, in this case, there is obviously no deceleration, and the universe will expand with a constant speed, $R(t) = H_0 t$.

Otherwise, the solution of Friedmann equation can be most conveniently written in parametric form: For $\Omega_0 > 1$:

$$R = a(1 - \cos\theta), t = b(\theta - \sin\theta),$$

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)}, b = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}.$$

Obviously, R will reach its max when $\theta = \pi$, and $t_{max} = \frac{\pi\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}$.

Similar solution can be found for the open model.

The relevance of the closed model solution is: as we will see later when we discuss galaxy formation, for a local area in the universe where the density is higher than the critical density, it can be treated like a closed universe, and it will collapse at this t_{max} timescale.

We can rewrite Friedmann equation:

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

by dividing $\dot{R}^2$ in both sides, and ask: $\Omega_\Lambda = \Lambda/3H_0^2$, and $\Omega_k = -kc^2/H_0^2$, thus:

$$\left[\frac{H(z)}{H_0}\right]^2 = \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda.$$

This is the form of Friedmann equation I find most useful, because it connects Hubble constant with other cosmological parameters: density, curvature and cosmological constant. Obviously, at $z = 0$:

$$1 = \Omega_m + \Omega_k + \Omega_\Lambda.$$

4.4 Λ model

The solution for Λ case is more complicated. But let's look at Friemann equation again, if $\Lambda \neq 0$,

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \kappa c^2 + \frac{1}{3}\Lambda R^2$$

There are three terms: the gravity term, curvature term and Λ term. See how different terms change with redshift. At $R >> 1$, the curvature term doesn't grow with R, and can be ignored; the gravity term goes slower than the Λ term, and can be ignored as well. So we will have that in the late stage of expansion in a Λ universe,

$$\dot{R} = \sqrt{\Lambda/3}R, R \propto \exp(\sqrt{\Lambda/3}t)$$

The universe is going to expand exponentially if $\Lambda > 0$! Note that this is true regardless of the curvature and as far as $\Lambda > 0$, even if it is relatively small, you can not escape the fate that the universe is going to expand exponentially , and it seems that we can't avoid this fate now. So very small (many billion years later), the universe is going to be exceedingly empty. We will meet this exponential expansion again twice in this class, one about the dark energy, the other about inflation.

4.5 flatness problem

So how about when R is very small, in other words, at high redshift? For matter, we have $\rho \propto R^{-3} \propto (1+z)^3$, this term is going to dominate the Friedmann equation at early epoch, since the gravity term will go as $(1+z)$, curvature term is a constant, and Λ term goes as $R^2 \propto (1+z)^{-2}$. If we ignore the last two terms, then

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2,$$

and divide this by R^2, and note that $\rho_c = 3H^2/8\pi G$, we have $\Omega = \rho/\rho_c = 1$ at high redshift, regardless of what the current day curvature and cosmological constant is. There are two things to learn from this:

1. at high-redshift, the universe can always be approximated as being an Einstein-de Sitter model. Given the range of cosmological parameters we know today, the universe can be regarded as more or less being flat at $z > 4$ or so.

2. what is called the flatness problem. That is: if the curvature of the universe we measure today is close to zero, or Ω_0 is within a factor of 10 from unity, then at the very beginning, the universe must have been fine tuned to be very close to completely flat. Because if there is any very small deviation from flat at high redshift, it will be blow up more or less by a factor of $1 + z$ now. The flatness of the universe is a strong function of z, so why we happen to be living at a time Ω_0 is not far from 1, while if the curvature is not zero, then Ω_0 can be equally possible to be 1000 or 1/1000? Come back to cosmological principle, this argues that if we believe that we are not living in a special time, then the universe must have always been precisely flat from the beginning to the end. This argument, or solution, for the flatness problem, is a strong motivation for inflation model, in fact, you can say that it is a prediction for such inflation model, because at that time, there was few evidence that the universe is flat.

4.6 Energy Density

Now let's consider the relation between the $\ddot{R}$ and Friedmann equations. Differentiating the Friedmann equation gives

$$\ddot{R} = \frac{8\pi G}{3}\rho R + \frac{4\pi G}{3}R^2\frac{\dot{\rho}}{\dot{R}} + \frac{1}{3}\Lambda R$$

Subtracting the original GR acceleration gives

$$R\frac{\dot{\rho}}{\dot{R}} = -3\left(\rho + \frac{p}{c^2}\right)$$

We can write this as

$$R\frac{d\rho}{dR} = -3\left(\rho + \frac{p}{c^2}\right)$$

where we consider ρ as a function of R.

We are familiar with normal matter. Here, ρ is the familiar density and $p \ll \rho c^2$. So $\rho \propto R^{-3}$ as we have used.

What about other kinds of matter? Photons have a pressure that is a third of their energy density. We write $p = \rho c^2/3$. That means that $\rho \propto R^{-4}$. This corresponds to the number density of photons dropping as R^{-3} while their frequencies and hence energies decrease as R^{-1}.

In the case when $\rho \propto (1+z)^4$, it is easy to show that

$$\dot{R}^2 \propto 1/R^2$$

or

$$R \propto t^{1/2}$$

We often separate ρ into two pieces, the non-relativistic matter and the relativistic photons. We define $\Omega_M = \rho_m/\rho_c$ and $\Omega_r = \rho_r/\rho_c$ so that

$$\frac{\rho}{\rho_c} = \Omega_m(1+z)^3 + \Omega_r(1+z)^4$$

Note that since radiation goes as $(1+z)^4$, at some point it will dominate, so at very high redshift, it is always radiation dominated, and you can ignore mass when solving the dynamical evolution of the universe, while at later time, it is always matter dominated, as it is now. But in CMB era, it is radiation dominated. It is very important to remember that for matter: $\rho \sim (1+z)^{-3}$, and for radiation: $\rho \sim (1+z)^{-4}$.

4.7 Deceleration Parameter

Let's introduce one more parameter. We showed the Hubble constant is the expansion rate of the universe. So how do we define the acceleration, or more precisely, the deceleration, of the universe. We define:

$$q_0 = -\left(\frac{R\ddot{R}}{\dot{R}^2}\right)_{t_0}.$$

Substituting this definition to basic equations, when without Λ, and without pressure term, it is immediately obvious that

$$q_0 = \Omega_0/2.$$

So this shows that the deceleration of expansion is directly related to the mass density. Not to be surprised, the deceleration is caused by the gravity. But what it gives us is another way to measure the density of the universe, i.e., we know the expansion rate of the universe today; if we can somehow measure the expansion rate of the universe at high-redshift, then we can measure the deceleration and thus the density. This flavor of measuring Ω is called the geometrical measurement.

Note that in the absence of Λ, the universe always decelerate. But the above relation is not valid in case of Λ. As you probably know, the universe is actually accelerate.

5 Observations and Cosmological Parameters

We discussed the observables in cosmology, such as the different kind of distance, the co-moving, angular diameter, and luminosity distances, the volume, and look-back time. We derive their basic relation with RW metric, but since at that time we didn't know R(t), we can't write down their relation with redshift. Now we can.

If we set $\Lambda = 0$, then we have

$$H(z) = H_0\sqrt{\Omega_0(1+z)^3 - \kappa c^2(1+z)^2} = H_0\sqrt{\Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2}$$

Let's use this to find our distance-redshift relations.

$$r = \int_0^z \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2}}$$

The integral is a bit of a mess and the expression depends on whether $1 - \Omega_0$ is positive or negative.

The next step is to calculate $D = R_c \sin(r/R_c)$ (or sinh), because we found that $D_A = D/(1+z)$ and $D_L = D(1+z)$. This yields

$$D = \frac{2c}{H_0 \Omega_0^2 (1+z)} \left\{ \Omega_0 z + (\Omega_0 - 2) \left[(\Omega_0 z + 1)^{1/2} - 1 \right] \right\}$$

for any Ω_0.

Hence, we have just found the relation between distances and redshift. For small z, we have $D = (c/H_0)\left[z - z^2(1+\Omega/2)/2 + \ldots\right]$

c/H_0 appears a lot in cosmology. It is known as the Hubble distance. It is $3h^{-1}$ Gpc. Astronomers like to write $H_0 = 100h$ km/s/Mpc.

What do these distance relations look like? Let's consider $\Omega = 1$. Here

$$D = (2c/H_0)\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

As $z \to \infty$, $D \to 2c/H_0$.

However, the angular diameter distance is $D/(1+z)$ so this actually goes to 0! The maximum D_A is at $z = 5/4$ and is $8c/27H_0$. The angular diameter distance for $0.5 < z < 5$ is always about $1h^{-1}$ Gpc. This means that $1''$ is about $5h^{-1}$ kpc (physical) at cosmological distances.

The luminosity distance, however, grows quickly with z. High-redshift objects may not get smaller, but they do get much fainter!

You can do the same for open cosmology or cosmology with a Λ. You need to use a lot of these calculations when working on cosmology, and when working on extragalactic astronomy in general. I recommend a few tools: (1) David Hogg's paper (astro-ph/9905116); (2) astropy.cosmology; (3) Ned Wright's cosmology calculator, http://www.astro.ucla.edu/ $wright$/CosmoCalc.h (4) iCosmos, http://www.icosmos.co.uk/; (5) Cosmology calculation i-phone app CosmoCalc, free.

What are these relations used for? (1) given a cosmology, and measured redshift, you can then derive the diameter, luminosity and time of a high-redshift source from the observed angular size, flux, and redshift.

(2) if you can get the diameter, luminosity etc. independently, then you can use these figures to constrain cosmology, like Hubble constant, density, cosmological constant etc.

We have introduced a number of cosmological parameters.

- $H_0 = \dot{R}/R|_0$, is the current expansion rate of the universe.

- t_0 is the current age of the universe since the BB.
- Ω_m is the current density parameter of the Universe.
- k is the curvature of the universe, deciding the geometry.
- Λ is the cosmological constant.
- q_0 is the deceleration parameter.

Clearly, they are not all independent. Indeed, they are tied by Friedmann equations. In matter dominated era, as the equation we showed above, the entire expansion history can be described by three parameters, including a scale (Hubble constant, or age of the universe), and two parameters that specify the relative contribution of matter (including dark matter), curvature and cosmological constant to the total energy-density budget of the universe. This is our **Robertson-Walker-Friedmann world model.** The most important task of a cosmologist is to understand what our world model is. And our biggest task is to test whether this world model, and which version of it, is supported by our observations. This is called the **cosmological tests.**

In fact, our cosmological test, or our cosmological model, includes even more parameters, because we are interested in not only the expansion history of the material dominated era, but (1) the state of the universe in radiation dominated era, i.e., CMB, (2) the growth of fluctuation in the universe.

Planck results show that our cosmology can be described by essentially 7 parameters that fit all existing observations satisfactorily. These observations are (almost all):

(1) the expansion history of the universe, including measurements such as supernova, which we will discuss next lecture;

(2) the anisotropy of the CMB;

(3) the large scale structure in low-z universe, including measurements of galaxy clustering and structure of the IGM, as well as clusters of galaxies

And all of them can be fit by seven parameters (paper said six, because it is assuming a flat geometry: $H_0 = 70$, $\Omega_m = 0.28$, $\Omega_b = 0.05$, $k = 0$, and then two parameters, the zero point and slope that specify the density fluctuation power spectrum, and one parameter describes the optical depth of free electrons to CMB that will affect CMB photons through Compton scattering. We will discuss Ω_b, and the other three later. For the moment, we will concentrate on the three parameters that appear in Friedmann equations, i.e., the expansion history of the universe, which we generally call the **classical cosmological tests**. But the cosmological tests will really dominate our discussions in this class in the next 1.5 months of the class. We will discuss how to measure Ω_b in the BBN lecture, and discuss CMB in details, as well as large scale structure tests in details later.

Next in our class, we will discuss (1) Hubble constant; (2) age; and (3) Ω_m very briefly, and (4) Λ and dark energy.