Assignment 1
Astronomy 541

While the problems that we assign for credit will generally be more exact calculations, we want to encourage you to practice your skills at “back of the envelope” calculations. In that spirit, let us suggest that you estimate answers to the following questions and that you and a study partner come up with similar questions to practice on. Note that some of these questions don’t necessarily have a single, simple answer; instead, you’ll need to recognize unspecified aspects and incorporate your assumptions along the way. [These questions assume some astronomical background. Your undergraduate extragalactic astronomy class will help to give you some basic numbers, if they are not covered by AST 540 yet, in any case, try them out, but don’t worry if you don’t have the background.] Note that the following are not being graded; they are here for your own practice.

1) If the protons locked up in the stars in galaxies were spread evenly throughout the universe, what would the density be? What would the number density be? How does this compare to the density in the interstellar medium of the Milky Way?

2) How does the rest-mass energy density of these protons compare to the energy density of the CMB, which is a blackbody of temperature 2.725 Kelvin?
Assignment: Due Due Thursday, Sep 11, in class

Problem 1 (5 pts): Show that the metric for a 3-sphere (i.e. a sphere in 4-dimensional space) of radius \( R_c \) is

\[
d\ell^2 = dr^2 + R_c^2 \sin^2(r/R_c) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]  

Hints: The 3-sphere can be written as the surface \( w^2 + x^2 + y^2 + z^2 = R_c^2 \) in 4-dimensional Cartesian space. This surface can be parameterized as

\[
\begin{align*}
w &= R_c \cos \chi \\
z &= R_c \sin \chi \cos \theta \\
x &= R_c \sin \chi \sin \theta \cos \phi \\
y &= R_c \sin \chi \sin \theta \sin \phi
\end{align*}
\]

The distance between two infinitesimally close points on the surface doesn’t depend on whether one measures in the 4-space or in the 3-dimensional manifold. So one can start with the Cartesian metric \( d\ell^2 = dw^2 + dx^2 + dy^2 + dz^2 \) and substitute the differentials based on the change of coordinate systems, e.g. \( dw = -R_c \sin \chi \, d\chi \). After simplifying this, change variables once more, setting \( r = R_c\chi \).

Problem 2 (5 pts): The metric of the homogeneous hyperbolic 3-space is

\[
d\ell^2 = dr^2 + R_c^2 \sinh^2(r/R_c) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\]

for a constant \( R_c \) (note the small change from equation 1).

Show that the substitution \( x = R_c \sinh(r/R_c) \) makes the metric

\[
d\ell^2 = \frac{dx^2}{1 - \kappa x^2} + x^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\]

where \( \kappa = -1/R_c^2 \).

Show that a similar substitution takes the spherical metric (1) to the form (2) but with \( \kappa = +1/R_c^2 \).

Noting that the flat metric is simply (2) with \( \kappa = 0 \), you can see that all three homogeneous metrics can be written in a single form.

In words, the difference between these coordinate systems is whether we label the radial direction by the distance along the radial spoke or the circumference of the circle (divided by 2\( \pi \)).

Problem 3 (10 pts):

The Hubble constant (defined as the ratio between physical velocities and physical distances) as a function of time is defined as

\[
H(t) = \frac{1}{R(t)} \frac{dR}{dt}
\]
Consider a flat cosmology in which the scale factor \( R(t) = (t/t_0)^{2/3} \) where \( t_0 \) is the age of the universe today. We will later find that this is the case of a critical density, matter-dominated universe. Compute:

a) The Hubble constant as a function of redshift.

b) The comoving distance (also known as the coordinate distance) between us and a given redshift.

c) The angular diameter distance at a given redshift, i.e. the radians subtended on the sky by an object spanning one unit of length transverse to our line-of-sight.

d) The luminosity distance at a given redshift, i.e. the distance needed to convert bolometric luminosity to bolometric flux.

e) The comoving volume per unit redshift per steradian at a given redshift.

f) The elapsed time between a given redshift and today.