Assignment 4

Astronomy 541

Assignment: Due Friday, Oct 26, in class

Problem 1 (5 pts): For $\Omega_m = 1$ and $c_s = 0$, show that the gravitational potential ϕ of a growing-mode perturbation is time independent (as usual, ignore the homogeneous term in the Poisson equation).

Argue that in a low-density universe, where the growth lags behind $D \propto R$ and eventually converges to a constant, the gravitational potential decays to zero.

Problem 2 (5 pts): In class, we derived the following differential equation for the evolution of the amplitude of small perturbations:

$$\frac{d^2D}{dt^2} + 2H\frac{dD}{dt} = \left(4\pi G\rho_h - \frac{c_s^2k^2}{R^2}\right)D$$

Remember that H and ρ_h (the average density of the universe) are functions of time.

a) For pressureless matter in an open universe with $\Lambda = 0$, the above equation has the solution

$$D(t) = 1 + \frac{3}{x} + \frac{3\sqrt{1+x}}{x^{3/2}}\ln\left[\sqrt{1+x} - \sqrt{x}\right],$$

where $x = R(t)(1-\Omega_m)/\Omega_m$ and $R = 1$ today. Here, Ω_m is the density of matter.

If $\Omega_m = 0.3$, by what factor has the amplitude of perturbations grown between $z = 99$ and today? Between $z = 1$ and today? Compare these results to the those in an $\Omega_m = 1$ universe.

b) For Λ cosmologies, the solution for the growth function requires special functions (elliptic functions or beta functions). However, there is a fitting formula (Carroll, Press, Turner 1992, adapted from Lahav et al 1991) that holds for matter-dominated universes with curvature and Λ. The formula says that the growth function at $z = 0$ relative to that at a large initial z is

$$\frac{D(0)}{D(z_i)(1+z_i)} = \frac{5\Omega_m}{2}\left[\Omega_m^{4/7} - \Omega_\Lambda + (1+\Omega_m/2)(1+\Omega_\Lambda/70)\right]^{-1}$$

At large z the growth function scales as $(1+z)^{-1}$, so we don't need to specify a particular z_i.

Using the formula, compute the factor by which the amplitude of structure has grown from $z = 99$ to $z = 0$ for a universe of $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Compare this to the open universe in part (a) and to the $\Omega_m = 1$ case. You might want to try checking your results from part (a) too!

For future reference, to apply this formula to get the growth at other redshifts, you have to rescale Ω_m and Ω_Λ to the value that an observer at that redshift would measure and then divide by $1+z$ to accomplish a rescaling of z_i. If that's confusing, consider the formula for $\Omega_m = 1$ and $\Omega_\Lambda = 0$ to understand the z_i correction and then consider that the formula is essentially saying how much a cosmology's growth lags that of Einstein-de Sitter given a common beginning.

Problem 3 (10 pts): The power spectra of cold dark matter cosmologies typically have $P(k) \propto k$ at small k and $P(k) \propto k^{-3}$ at large k. For simplicity, we will adopt the form (not completely accurate!)

$$P(k) = \frac{Ak}{(1 + k^2 s^2)^2} \tag{1}$$

where s is the break radius, which we'll choose to be $20h^{-1}$ Mpc. A is a (as yet arbitrary) normalization. Note: this power spectrum only holds at early times when the perturbations are small. It is therefore called the "linear-regime" power spectrum. At late times, non-linear evolution will alter this spectrum.

In class, we derived that the mean square variation in a region defined by a window function $W(r)$ is

$$\sigma^2 = \frac{1}{2\pi^2} \int_0^\infty dk\, k^2 P(k) \left|\hat{W}(k)\right|^2 \tag{2}$$

where $\hat{W}$ is the Fourier transform of W and where $W(r)$ is normalized to have unit integral over all space, i.e. $\int d^3r\, W(r) = 1$. Note that the last assertion implies that $\hat{W}(0) = 1$.

a) What are the dimensions of P and A?

b) At what k is P maximized?

c) In class, we used a window that was non-zero and constant only inside a radius R. This is called a spherical tophat. The Fourier transform of this window was $3j_1(kR)/kR$. Unfortunately, computing σ for a given R with this window requires a numerical integration.

Instead, consider a window whose Fourier transform is unity inside of a radius K (and zero outside). In cosmology, this is called a "sharp k-space" filter, but everyone else would just call it a low-pass filter. The real-space $W(r)$ is a little messy—it rings and damps only slowly—but it still has a typical scale, which is approximately $R = \pi/2K$. The coefficient here seems strange (why not $2\pi/R$ or $1/R$?), but there is an odd logic: the diameter of the sphere is $2R$ and the mean square overdensity should be dominated by waves that have a half wavelength equal to this diameter.

For this filter, compute σ as a function of K and hence of R.

Compute the limits of σ for small and large K. Does the answer to part (b) look special, e.g. does it correspond to a maximum in σ?

d) Normalize (i.e. find A) the power spectrum by forcing $\sigma = 0.9$ for $R = 8h^{-1}$ Mpc. Note: Cosmologist very often refer to σ_8, which is nearly this quantity, but computed with the real-space spherical tophat window.

What is the value of σ for $R = 0.8h^{-1}$ Mpc? For $R = 80h^{-1}$ Mpc?

e) Length scales may not mean much to you yet, so instead let's convert to mass scales by using $M = (4\pi/3)\rho_c \Omega_m R^3$. Consider $\Omega_m = 0.3$ and $h = 0.7$. Compare the scales in parts (d) and (e) to those of galaxies and clusters.