Assignment 4  
Astronomy 541

Assignment: Due Tuesday, Oct 28, in class

Problem 1 (5 pts): For $\Omega_m = 1$ and $c_s = 0$, show that the gravitational potential $\phi$ of a growing-mode perturbation is time independent (as usual, ignore the homogeneous term in the Poisson equation).

Argue that in a low-density universe, where the growth lags behind $D \propto R$ and eventually converges to a constant, the gravitational potential decays to zero.

Problem 2 (5 pts): In class, we derived the following differential equation for the evolution of the amplitude of small perturbations:

$$\frac{d^2 D}{dt^2} + 2H \frac{dD}{dt} = \left(4\pi G \rho_h - \frac{c_s^2 k^2}{R^2}\right) D$$

Remember that $H$ and $\rho_h$ (the average density of the universe) are functions of time.

a) For pressureless matter in an open universe with $\Lambda = 0$, the above equation has the solution

$$D(t) = 1 + \frac{3}{x} + \frac{3\sqrt{1+x}}{x^{3/2}} \ln \left[\sqrt{1+x} - \sqrt{x}\right],$$

where $x = R(t)(1 - \Omega_m)/\Omega_m$ and $R = 1$ today. Here, $\Omega_m$ is the density of matter.

If $\Omega_m = 0.3$, by what factor has the amplitude of perturbations grown between $z = 99$ and today? Between $z = 1$ and today? Compare these results to the those in an $\Omega_m = 1$ universe.

b) For $\Lambda$ cosmologies, the solution for the growth function requires special functions (elliptic functions or beta functions). However, there is a fitting formula (Carroll, Press, Turner 1992, adapted from Lahav et al 1991) that holds for matter-dominated universes with curvature and $\Lambda$.

The formula says that the growth function at $z = 0$ relative to that at a large initial $z$ is

$$\frac{D(0)}{D(z_i)(1 + z_i)} = \frac{5\Omega_m}{2} \left[\Omega^{4/7}_m - \Omega + (1 + \Omega_m/2)(1 + \Omega/70)\right]^{-1}$$

At large $z$ the growth function scales as $(1 + z)^{-1}$, so we don’t need to specify a particular $z_i$.

Using the formula, compute the factor by which the amplitude of structure has grown from $z = 99$ to $z = 0$ for a universe of $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Compare this to the open universe in part (a) and to the $\Omega_m = 1$ case. You might want to try checking your results from part (a) too!

For future reference, to apply this formula to get the growth at other redshifts, you have to rescale $\Omega_m$ and $\Omega_\Lambda$ to the value that an observer at that redshift would measure and then divide by $1 + z$ to accomplish a rescaling of $z_i$. If that’s confusing, consider the formula for $\Omega_m = 1$ and $\Omega_\Lambda = 0$ to understand the $z_i$ correction and then consider that the formula is essentially saying how much a cosmology’s growth lags that of Einstein-de Sitter given a common beginning.
Problem 3 (10 pts): The power spectra of cold dark matter cosmologies typically have \( P(k) \propto k \) at small \( k \) and \( P(k) \propto k^{-3} \) at large \( k \). For simplicity, we will adopt the form (not completely accurate!)

\[
P(k) = \frac{Ak}{(1 + k^2 s^2)^2}
\]

where \( s \) is the break radius, which we’ll choose to be \( 20h^{-1} \) Mpc. \( A \) is a (as yet arbitrary) normalization. Note: this power spectrum only holds at early times when the perturbations are small. It is therefore called the “linear-regime” power spectrum. At late times, non-linear evolution will alter this spectrum.

In class, we derived that the mean square variation in a region defined by a window function \( W(r) \) is

\[
\sigma^2 = \frac{1}{2\pi^2} \int_0^\infty dk \ k^2 P(k) \left| \hat{W}(k) \right|^2
\]

where \( \hat{W} \) is the Fourier transform of \( W \) and where \( W(r) \) is normalized to have unit integral over all space, i.e. \( \int d^3r \ W(r) = 1 \). Note that the last assertion implies that \( \hat{W}(0) = 1 \).

a) What are the dimensions of \( P \) and \( A \)?

b) At what \( k \) is \( P \) maximized?

c) In class, we used a window that was non-zero and constant only inside a radius \( R \). This is called a spherical tophat. The Fourier transform of this window was \( 3j_1(kR)/kR \). Unfortunately, computing \( \sigma \) for a given \( R \) with this window requires a numerical integration.

Instead, consider a window whose Fourier transform is unity inside of a radius \( K \) (and zero outside). In cosmology, this is called a “sharp \( k \)-space” filter, but everyone else would just call it a low-pass filter. The real-space \( W(r) \) is a little messy—it rings and damps only slowly—but it still has a typical scale, which is approximately \( R = \pi/2K \). The coefficient here seems strange (why not \( 2\pi/R \) or \( 1/R \)?) but there is an odd logic: the diameter of the sphere is \( 2R \) and the mean square overdensity should be dominated by waves that have a half wavelength equal to this diameter.

For this filter, compute \( \sigma \) as a function of \( K \) and hence of \( R \).

Compute the limits of \( \sigma \) for small and large \( K \). Does the answer to part (b) look special, e.g. does it correspond to a maximum in \( \sigma \)?

d) Normalize (i.e. find \( A \)) the power spectrum by forcing \( \sigma = 0.9 \) for \( R = 8h^{-1} \) Mpc. Note: Cosmologist very often refer to \( \sigma_8 \), which is nearly this quantity, but computed with the real-space spherical tophat window.

What is the value of \( \sigma \) for \( R = 0.8h^{-1} \) Mpc? For \( R = 80h^{-1} \) Mpc?

e) Length scales may not mean much to you yet, so instead let’s convert to mass scales by using \( M = (4\pi/3)\rho_c \Omega_m R^3 \). Consider \( \Omega_m = 0.3 \) and \( h = 0.7 \). Compare the scales in parts (d) and (e) to those of galaxies and clusters.