Solutions for Assignment 3

Astronomy 541

Problem 1 (5 pts): In a universe dominated by radiation, we can write the Hubble constant as $H(z) = H_0\sqrt{\Omega_r(1+z)^4}$. Strictly speaking, if the universe if flat and has only radiation, then we have $\Omega_r = 1$, but it is useful to include this parameter, since it describes a universe dominated by radiation at early times while permitting us to connect the results to a *current* Hubble constant and radiation density.

With this, we have

$$t(z) = \int_z^\infty \frac{dz}{(1+z)H(z)} = \frac{1}{H_0\sqrt{\Omega_r}}\int_z^\infty \frac{dz}{(1+z)^3} = \frac{1}{H_0\sqrt{\Omega_r}}\frac{1}{2(1+z)^2} = \frac{1}{2H(z)} \tag{1}$$

and

$$r_H(z) = \int_z^\infty \frac{c\,dz}{H(z)} = \frac{c}{H_0\sqrt{\Omega_r}}\int_z^\infty \frac{dz}{(1+z)^2} = \frac{c}{H_0\sqrt{\Omega_r}}\frac{1}{(1+z)} = \frac{c(1+z)}{H(z)} = 2ct(z)(1+z) \tag{2}$$

The only differences between these formulae and our earlier formulae are the limits of integration.

Note the scalings of $r_H(z)$: ct is the physical distance travelled, $1+z$ corrects this to a comoving distance assuming that all of the travelling was done at the final redshift z, and a factor of 2 corrects for the fact that travel at earlier times was slightly more efficient in terms of comoving distance.

Problem 2 (5 pts): a) Now we use $H(z) = H_0\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3\right]^{1/2}$. The integral can be done by changing variables to $R = (1+z)^{-1}$. $dR = -dz/(1+z)^2$ or $dz = -dR/R^2$. It is useful to define the epoch of matter-radiation equality as $1+z_{eq} = \Omega_m/\Omega_r$.

For the comoving distance, we have

$$\begin{aligned} r_H(z) &= \int_z^\infty \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^{(1+z)^{-1}} \frac{dR}{R^2\sqrt{\Omega_r R^{-4} + \Omega_m R^{-3}}} = \frac{c}{H_0}\int_0^{(1+z)^{-1}} \frac{dR}{\sqrt{\Omega_r + \Omega_m R}} && (3)\\ &= \frac{2c}{H_0\Omega_m}\left[\sqrt{\Omega_r + \Omega_m R}\right]_0^{(1+z)^{-1}} = \frac{2c}{H_0\sqrt{\Omega_m}}\left[\sqrt{\frac{\Omega_r}{\Omega_m} + \frac{1}{1+z}} - \sqrt{\frac{\Omega_r}{\Omega_m}}\right] && (4)\\ &= \frac{2c}{H_0\sqrt{\Omega_m}\sqrt{1+z_{eq}}}\left[\sqrt{1+y} - 1\right] && (5) \end{aligned}$$

where $y = (1+z_{eq})/(1+z) \propto R$.

For early times with $y \ll 1$, the term in square brackets is $y/2$, which yields

$$r_H(z) \to \frac{c\sqrt{1+z_{eq}}}{H_0\sqrt{\Omega_m}(1+z)} = \frac{c(1+z)}{H_0\sqrt{\Omega_r(1+z)^2}} = \frac{c(1+z)}{H(z)} \tag{6}$$

as in Problem 1.

b) We use $\Omega_r h^2 = 4.2 \times 10^{-5}$ and $\Omega_m h^2 = 0.147$, so $1 + z_{eq} = 3500$. At $z = 1000$, we have $y = 3.5$, which makes $r_H = 0.069c/H_0 = 208h^{-1}$ Mpc.

At $z = 3500$, we have $y = 1$, which makes $r_H = 0.026c/H_0 = 77h^{-1}$ Mpc.

Obviously these comoving distances are much smaller than the current size of the observable universe!

Problem 3 (5 pts): The optical depth can be computed as an integral along the line of sight of the cross-section times the density of free electrons.

$$\tau = \int d\ell \; \sigma_T n_e = \int dt \; c\sigma_T n_e = \int \frac{c\,dz}{(1+z)H(z)} \sigma_T n_e \tag{7}$$

We will use $H(z) = H_0\sqrt{\Omega_m(1+z)^3}$. I had only required $\Omega_m = 1$, but this form allows us to compute the optical depth at $z \gg 1$ for other cosmologies.

The number density of free electrons will scale as $(1+z)^3$. The present day value is

$$n_{e,0} \approx \frac{\rho_b}{m_p} = \frac{\Omega_b \rho_{crit}}{m_p} = \Omega_b h^2 \frac{1.88 \times 10^{-29} \text{ g cm}^{-3}}{m_p} = 1.12 \times 10^{-5} \Omega_b h^2 \text{ cm}^{-3} \tag{8}$$

where m_p is the mass of the proton.

The optical depth is then

$$\begin{aligned} \tau &= \int_0^z dz \frac{c\sigma_T n_{e,0}(1+z)^3}{H_0\sqrt{\Omega_m}(1+z)^{5/2}} = \frac{2c\sigma_T n_{e,0}}{3H_0\sqrt{\Omega_m}} \left[(1+z)^{3/2} - 1\right] && (9) \\ &= 9.2 \times 10^{-4} h^{-1} (1+z)^{3/2} \Omega_m^{-1/2} \left(\frac{\Omega_b h^2}{0.02}\right) && (10) \end{aligned}$$

where the last line assumes $z \gg 1$ (consistent with our $H(z)$ approximation).

If $h = 0.7$ and $\Omega_m = 1$, then τ will reach unity at $z = 82$.

In fact, we expect that the universe becomes reionized at $z = 10$ or 20. For such redshifts, the optical depth is substantially less than 1, but it is still noticeably non-zero ($\tau \approx 0.15$). The WMAP satellite claims to see a signature of this optical depth!

Problem 4 (5 pts): a) If X decouples when the universe is much hotter than $m_X c^2$, then it will be in a ultrarelativistic thermal distribution. Since X is a boson and there are two spin states (X and $\bar{X}$), the number density will be the same as the photons. Today, that is 411 cm^{-3}.

$\Omega_X = \rho_X/\rho_c = m_X n_X/\rho_c$, so $\Omega_X h^2 = 4 \times 10^7 (m_X c^2/1 \text{ GeV})$! This is vastly more than is observed.

An additional (and optional) refinement is to include the extra heating of the photons since the universe had a temperature of $\gg 1$ GeV. We know that there is a factor of $(4/11)^{1/3}$ getting back to a few MeV. At that time, $g_* = 10.75$, 2 for photons, $2 \times 2 \times (7/8)$ for electrons, and $6 \times 1 \times (7/8)$ for neutrinos. At $\gg 1$ GeV, we would have additional factors of $4(7/8)$ each for the muon and tau leptons, 8 for gluons, and $2 \times 2 \times 3 \times (7/8)$ for each quark (say, 5 neglecting the top quark; here we have 2 spins and 3 colors plus antiquarks), so $g_* \approx 80$. Hence, the annihilation of these species between 1 MeV and 10 GeV would increase the temperature at late times by another factor of $\sim 8^{1/3}$. Taking these together, we would predict that the number of X particles is suppressed by a factor of 20. At $\gg 100$ GeV, we would have additional terms for the W and Z bosons, the top quark, and perhaps the Higgs sector, further reducing the X abundance.