Solutions for Assignment 6

Astronomy 541

Problem 1 (10 pts): a) First, the comoving matter density is $\Omega_m \rho_{crit} = 8.3 \times 10^{10} h^2 M_\odot$ Mpc^{-3}. A sphere of radius $8h^{-1}$ Mpc has a mass of $1.8 \times 10^{14} h^{-1} M_\odot$. Hence, σ on the $5 \times 10^{14} h^{-1} M_\odot$ scale will be $\sigma_8 (5/1.8)^{-(3-1.5)/6)} = 0.62$. This makes $\nu = 2.73$. The density of clusters per logarithmic mass can then be computed with the formula to be $2 \times 10^{-6} h^3$ Mpc^{-3}.

b) If the normalization of the power spectrum drops by 20%, then σ_8 and σ_{5e14} will be lower by 10%. This makes $\nu = 3.04$. The number of clusters will be $1.0 \times 10^{-6} h^3$ Mpc^{-3}, a drop of 40%.

In other words, it doesn't take a very precise measurement of the abundance of galaxy clusters to get a good constraint on the power spectrum amplitude.

c) If we now switch to Einstein-de Sitter and want to keep the number of clusters of this mass the same, then we will need to adjust ν because the pre-factor ρ_m is larger. One finds that one needs $\nu = 3.19$ to make the same number of clusters, implying $\sigma = 0.53$ on the cluster scale.

Next, a sphere of radius $8h^{-1}$ Mpc now contains $6.0 \times 10^{14} h^{-1} M_\odot$. $\sigma = \sigma_8 (5/6)^{-1/4}$, so we find that $\sigma_8 = 0.55$.

d) At higher redshift, the values of σ are changed by the growth function. In ΛCDM, we have $\nu = 4.45$, so the comoving number density is $7 times 10^{-9} h^3$ Mpc^{-3}. This is small, but there's about $10^{10} h^{-3}$ Mpc^3 per steradian in the universe to that redshift, so one would find one of these clusters every 20 square degrees or so.

In Einstein-de Sitter, we have $\nu = 3.05 * 2 = 6.4$. This makes the comoving number density $1 \times 10^{-12} h^3$ Mpc^{-3}, about 10^4 times lower than the ΛCDM case! One predicts no massive clusters at high redshift in this model!

The arguments in this problem have been important in cosmological parameter estimation. Einstein-de Sitter models require a normalization of σ_8 at $z = 0$ that is fairly low. Galaxies are observed to have $\sigma_{8,gal} \approx 1$, so they must be highly biased in this model ($b \approx 2$). The observation of massive clusters at high redshift is very difficult to explain with $\Omega = 1$. However, the Achilles heel of this method is that one must have an accurate estimate of the mass of the clusters in one's sample. The exponential sensitivity of the numbers to σ also means that one is exponentially sensitive to errors in the mass measurements. This is particularly a problem at high redshift where the observations are more limited. For example, the mass function dn/dM is so steep at the high mass end that the largest clusters are very likely to be lower mass objects that have scattered up in their mass estimates by bad luck or some astrophysical variance.

Problem 2 (10 pts): a) The comoving volume per steradian per unit redshift is $dV/d\Omega\, dz = (c/H)S[r(z)]^2 = (c/H)r^2$. At $z = 3$, $r = 1.48c/H_0$ and $H = 4.46H_0$ in this cosmology. Hence, we have $1.3 \times 10^{10} h^{-3}$ Mpc^3 per steradian per unit redshift, which is $560h^{-3}$ Mpc^3 per square arcminute per half redshift. Hence, the comoving number density of these galaxies is $7 \times 10^{-4} h^3$ Mpc^{-3}.

b) Taking the above density as $dn/d\log(M)$ and using $n_{\rm eff} = -2$, we have

$$M = 1.6 \times 10^{13} h^{-1} M_\odot \nu e^{-\nu^2/2} \quad (1)$$

For ν, we use that $\sigma = 2.3/3.13 = 0.73$ on the $10^{12}h^{-1}M_\odot$ mass scale at $z = 3$. Scaling from that mass with $n_{\rm eff}$ gives $\sigma = 0.73(M/10^{12}h^{-1}M_\odot)^{-1/6}$ and $\nu = 2.32(M/10^{12}h^{-1}M_\odot)^{1/6}$. Inserting this into (1) yields a transcendental equation for M. One can either solve this by iteration or by simple root-finding. The result is $M = 1.67 \times 10^{12}h^{-1}M_\odot$ and $\nu = 2.53$.

Unfortunately, simple iteration is unstable for these parameters (although it is stable for masses below about $7 \times 10^{11}h^{-1}M_\odot$, e.g. part d). The mass at each step oscillates on either side of the true solution but with an increasing amplitude. A general trick in these circumstances for getting (or improving) convergence is to take two iteration steps and use the average of them for the next double step. This changes the series from an oscillating divergence to a monotonic convergence.

c) We predict $\sigma_8 = 0.85/3.13 = 0.27$ for the mass, but we observe $\sigma_8 \approx 1$ for the galaxies. This implies a bias of 3.7! That in turn requires $\nu = 2.36$ in the simple bias model. That value is achieved for halos of $1.1 \times 10^{12}h^{-1}M_\odot$.

It is counted as a success of the theory that these mass estimates agree to a factor of two!

d) If the duty cycle of having a detectable galaxy was only 1%, then we would require a density of halos of $0.07h^3$ Mpc^{-3}. That occurs in halos of mass $7.7 \times 10^{10}h^{-1}M_\odot$, which have $\nu = 1.51$. That means that the bias would only be 1.76, which would make $\sigma_{8,gal} = 0.48$, just half of what is observed.

This is considered to be strong evidence that the Lyman-break galaxies do not typically reside in halos of $10^{11}M_\odot$.

Problem 3:

a) $\frac{3}{2}k_BT = GM_{200}\mu m_H/r_{200}$. Using $M_{200} = \frac{4\pi}{3}200\rho_0 r_{200}^3$, we get $T = (\frac{4\pi}{3}200\rho_0)^{1/3}\frac{2G\mu m_H}{3k_B}M_{200}^{2/3}$. $\mu = 0.59$ for a fully ionized primordial plasma, and $\rho_0 = 2.76 \times 10^{-30}$g cm^{-3} for $\Omega = 0.3, h = 0.7$. This gives $T = 6.7 \times 10^7(M_{200}/10^{15}M_\odot)^{2/3}$ K.

b) The derivation in 1(c) is helpful here – the mass at r_{200} is equal to M_{200}:

$$M_{200} = 4\pi\int_0^{r_{200}} dr\; r^2\rho = 4\pi\rho_s r_s^3\int_0^{r_{200}/r_s} dx\;\frac{x}{(x+1)^2} = 4\pi\rho_s r_s^3\left[\log\left(1+\frac{r_{200}}{r_s}\right) - \frac{r_{200}}{r_{200}+r_s}\right] \quad (2)$$

Using $c \equiv r_{200}/r_s$ and $M_{200}/r_{200}^3 = (4\pi/3)200\rho_0$, we get

$$\rho_s = \frac{200c^3\rho_0}{3}\left[\log(1+c) - \frac{c}{c+1}\right]^{-1} \quad (3)$$

At $r = r_s$, $\rho(r_s) = \rho_s/4$. Electron density is simply $n_{e,s} = \rho(r_s) \times (\Omega_b/\Omega_m)/m_p$. Which is 1.7×10^{-3} cm^{-3} for c=10. **Note that you need to correct for the baryon fraction here. Most of the cluster is dark matter!** Note that this is independent of halo mass!

c) The cooling radius is defined as the radius r_c where the cooling time t_c equals the Hubble time t_H (which we will take as 13.7 Gyr). Hence for an NFW gas profile,

$$t_c(r) = kT/n_e(r)\Lambda = \frac{kT}{\Lambda}\frac{r(r+r_s)^2}{n_{e,s}r_s^3}. \quad (4)$$

We can equate the LHS to t_H, and rewrite the RHS with modest algebra as

$$t_H = \frac{kT}{\Lambda}\frac{c^3}{n_{e,s}}\frac{r_c}{r_{200}}\left(\frac{r_c}{r_{200}} + \frac{1}{c}\right)^2 \quad (5)$$

d) We now solve this for a $10^{15} M_\odot$, $c = 10$ halo. The virial temperature is $T = 6.5 \times 10^7$ K from part (a). For that temperature, $\Lambda = 1.37 \times 10^{-23}$ erg cm^3 s^{-1}. Using $n_{e,s}$ from above, we have

$$\frac{r_c}{r_{200}} \left(\frac{r_c}{r_{200}} + 0.1 \right)^2 = 0.4. \qquad (6)$$

We then solve this by trial-and-error to obtain $r_c \approx 0.1 r_{200}$.

e) For the $10^{14} M_\odot$ halo, $T = 1.4 \times 10^7$ K, and $\Lambda = 6.4 \times 10^{-24}$ erg cm^3 s^{-1}. Again by trial and error we obtain $r_c \approx 0.15 r_{200}$. This halo will have cooled out more of its material. Hence one expects more baryons in a "cool" form (i.e. stars or cold gas) as one goes to smaller masses; this is qualitatively consistent with observations.

Problem 4 (10 pts): a) The number of recombinations per unit volume is $\alpha n_e n_i$, while the number of ionizations per unit volume is Γn_H. In ionization equilibrium, these two must be equal. If the medium is nearly ionized, then $n_e = n_i \approx n_b$, the total density of baryons. This means that $n_H = \alpha n_b^2 / \Gamma$. The neutral fraction $x_H = n_H / n_b = \alpha n_b / \Gamma$ and hence depends linearly on the gas density and inversely on incident UV flux.

b) The hydrogen density today is 2.0×10^{-7} cm^3 given $\Omega_b h^2 = 0.024$ and a hydrogen fraction of 75%. At $z = 3$, it is 64 times higher. For $J_H = 0.3$, we have $\Gamma = 1.3 \times 10^{-12}$ s^{-1}. For $T = 15,000$ K, we have $\alpha = 3.2 \times 10^{-13}$ cm^3 s^{-1}. This gives a neutral fraction of 3.15×10^{-6}, which is very small!

c) The HI optical depth is $\tau_{HI} \propto \int \sigma n_{HI} dl \propto \int \frac{n_H^2}{\Gamma} \frac{dl}{dz} dz$, where l is the physical path length. To relate to comoving path length S, $\frac{dl}{dz} = \frac{dS/dz}{1+z} = \frac{c}{H(1+z)} \propto (1+z)^{-5/2}$ using $H(z) \propto (1+z)^{3/2}$ in the matter-dominated case. Meanwhile, $n_H \propto (1+z)^3$, so if $\Gamma =$constant, then $\tau_{HI} \propto \int (1+z)^{3.5} d(1+z) \propto (1+z)^{4.5}$. Transmission is $T = 1 - e^{-\tau} \approx \tau$ for $\tau \ll 1$ (which is true at the mean density for $z \ll 6$), therefore it also scales as $(1+z)^{9/2}$. FYI, the observed scaling is close to this, $\bar{T} \propto (1+z)^{4.3}$.

d) The mean free path is $1/\sigma n_H = 1/\sigma x_H n_b$, where $\sigma = 6.3 \times 10^{-18} (\nu_1/\nu)^{2.8}$ cm^2. Scaling from part (b), the neutral fraction will be 24 times higher, which is 7.6×10^{-5}. The mean hydrogen density at $z = 7$ is 10^{-4} cm-3. Hence the mean free path is then $2.1 \times 10^{25} (\nu/\nu_1)^{2.8}$ cm, which is $6.4 (\nu/\nu_1)^{2.8}$ Mpc. The Hubble distance is $c/H(z) \approx c/H_0 \sqrt{\Omega_m (1+z)^3} = 354$ Mpc for $\Omega_m h^2 = 0.14$ and $z = 7$. These are equal for $\nu = 4.2 \nu_1 = 57$ eV.

In practice, 54.4 eV is the ionization energy for He^+, and we believe that most of the helium at $z = 7$ is only singly ionized, so there will be an enormous additional component of optical depth at energies above 54.4 eV (this is called the HeII Gunn-Peterson effect). There would also be a contribution from neutral He absorption at 26 eV. In other words, the numbers in this problem would leave the universe at least marginally optically thick across the far-UV.