Solutions for Assignment 8

Astronomy 541

Problem 1 (7 pts): a) The number of recombinations per unit volume is \( \alpha n_e n_i \), while the number of ionizations per unit volume is \( \Gamma n_H \). In ionization equilibrium, these two must be equal. If the medium is nearly ionized, then \( n_e = n_i \approx n_b \), the total density of baryons. This means that \( n_H = \alpha n_b^2 / \Gamma \). The neutral fraction \( x_H = n_H / n_b = \alpha n_b / \Gamma \) and hence depends linearly on the gas density and inversely on incident UV flux.

b) The hydrogen density today is \( 2.0 \times 10^{-7} \) cm\(^3\) given \( \Omega_b h^2 = 0.024 \) and a hydrogen fraction of 75%. At \( z = 3 \), it is 64 times higher. For \( J_H = 0.3 \), we have \( \Gamma = 1.3 \times 10^{-12} \) s\(^{-1}\). For \( T = 15,000 \) K, we have \( \alpha = 3.2 \times 10^{-13} \) cm\(^3\) s\(^{-1}\). This gives a neutral fraction of \( 3.15 \times 10^{-6} \), which is very small!

c) The HI optical depth is \( \tau_{HI} \propto \int \sigma n_H dl \propto \int n_b^2 dl dz \), where \( l \) is the physical path length. To relate to comoving path length \( S, \frac{dl}{dz} = \frac{dS}{dz} = \frac{c}{H(z)(1+z)} \propto (1+z)^{-5/2} \) using \( H(z) \propto (1+z)^{3/2} \) in the matter-dominated case. Meanwhile, \( n_H \propto (1+z)^3 \), so if \( \Gamma \) = constant, then \( \tau_{HI} \propto \int (1+z)^{3.5} dl(1+z) \propto (1+z)^{4.5} \). Transmission is \( T = 1 - e^{-\tau} \approx \tau \) for \( \tau \ll 1 \) (which is true at the mean density for \( z \ll 6 \)), therefore it also scales as \( (1+z)^{9/2} \). FYI, the observed scaling is close to this, \( T \propto (1+z)^{4.3} \).

d) The mean free path is \( 1/\sigma n_H = 1/\sigma x_H n_b \), where \( \sigma = 6.3 \times 10^{-18} (\nu/\nu_1)^{2.8} \) cm\(^2\). Scaling from part (b), the neutral fraction will be 24 times higher, which is \( 7.6 \times 10^{-5} \). The mean hydrogen density at \( z = 7 \) is \( 10^{-4} \) cm\(^{-3}\). Hence the mean free path is then \( 2.1 \times 10^{25} (\nu/\nu_1)^{2.8} \) cm, which is 6.4(\( \nu/\nu_1 \))\(^{2.8}\) Mpc. The Hubble distance is \( c/H(z) \approx c/H_0 \sqrt{\Omega_m (1+z)^3} = 354 \) Mpc for \( \Omega_m h^2 = 0.14 \) and \( z = 7 \). These are equal for \( \nu = 4.2 \nu_1 = 57 \) eV.

In practice, 54.4 eV is the ionization energy for \( He^+ \), and we believe that most of the helium at \( z = 7 \) is only singly ionized, so there will be an enormous additional component of optical depth at energies above 54.4 eV (this is called the HeII Gunn-Peterson effect). There would also be a contribution from neutral He absorption at 26 eV. In other words, the numbers in this problem would leave the universe at least marginally optically thick across the far-UV.

Problem 2 (7 pts): a) If sources are distributed uniformly and neglecting cosmological effects, the number of sources in a spherical shell scales as \( r^2 dr \). The luminosity from these sources is down by \( 1/r^2 \), so the total received flux scales as \( dr \). Hence, the distant sources dominate the integrated flux.

When one includes cosmological effects, the integral is cut off at roughly the Hubble distance. But it’s still true that distant sources still dominate.

b) The optical depth scales inversely with the incident UV radiation at a particular location. The UV radition is the background plus that of the quasar, hence \( \Gamma_{HI} \propto J_H + L_H/4\pi r^2 \).

However, this isn’t quite right: \( J_H \) should be defined as the specific intensity (I didn’t state this correctly in the problem set), which is flux per steradian, whereas \( L_H/4\pi r^2 \) is just a flux. The hydrogen atoms only care about the local number density of photons, so we should divide \( L_H/4\pi r^2 \)
by another factor of $4\pi$ to convert it to an “effective isotropic background” for the comparison. I didn’t grade on this point, though some of you did get it correct. Moral: Watch your units!

Since $\tau \propto \Gamma^{-1}_{HI}$, we have $\tau_{\text{nearby}} = J_H/[J_H + L_H/(4\pi r)^2] = [1 + L_H/J_H(4\pi r)^2]^{-1}$. Note that the quasar flux is in addition to the uniform background $J_H$, which is still present near the quasar!

c) The optical depth is reduced by a factor of two at the radius $r_0 = \sqrt{L_H/J_H}/4\pi$. With a bandwidth of $10^{15}$ Hz, $J_H = 10^{-21}$ ergs cm$^{-2}$ s$^{-1}$ sr$^{-1}$ Hz$^{-1}$ becomes $10^{-6}$ ergs cm$^{-2}$ s$^{-1}$ sr$^{-1}$. Hence, $r_0 = 4.7$ Mpc (using $J_{21} = 0.3$).

The nonlinear mass scale is about $10^{12} M_\odot$, which is also around 1.4 comoving Mpc (using $M = 4\pi r^3/3$). Hence matter clustering, if not accounted for, causes an underestimate of the proximity effect (because there would be more matter there than expected from homogeneity and hence more absorption), which would cause an overestimate of $J_H$. This is further exacerbated if quasars don’t live in typical halos, but rather live in the most massive halos (with stronger clustering), as is now believed. This is now thought to be the reason why early proximity effect calculations yielded $J_H$ values $\sim 3 - 10 \times$ higher than direct quasar counting estimates.

**Problem 3 (6 pts):**

a) Specific KE= $\frac{1}{2}b^2$; specific thermal energy= $k_B T/m_H$. Equating gives $b = \sqrt{\frac{2k_B T}{m_H}} = 12.8$ km/s.

b) $R \equiv \Delta \lambda/\lambda = c/\Delta v$ using the first order Doppler formula. For $\Delta v = b = 12.8$ km/s, this gives $R \approx 23,000$. This shows why Echelle spectrographs are critical for accurately characterizing the Lyman alpha forest.

c) Line widths are a dispersion, hence add in quadrature. So $b_{\text{Hubble}} = \sqrt{b_{\text{total}}^2 - b_{\text{thermal}}^2} = 21.5$ km/s.

d) At the cosmic mean density, gas is expanding exactly with Hubble flow. For $\Omega_m = 1$, $H(z) = H_0(1 + z)^{3/2}$, so $H = 560$ km/s/Mpc. The width of 21.5 km/s then corresponds to 38 kpc (physical). If the regions was overdense, it would be contracting w.r.t. Hubble flow, so the effective $H$ would be smaller, and the inferred line-of-sight extent would be larger.