Solutions for Assignment 3

Astronomy 541

Problem 1 (5 pts): In a universe dominated by radiation, we can write the Hubble constant as \( H(z) = H_0 \sqrt{\Omega_r (1 + z)^2} \). Strictly speaking, if the universe if flat and has only radiation, then we have \( \Omega_r = 1 \), but it is useful to include this parameter, since it describes a universe dominated by radiation at early times while permitting us to connect the results to a current Hubble constant and radiation density.

With this, we have

\[
t(z) = \int_z^\infty \frac{dz}{(1 + z)H(z)} = \frac{1}{H_0 \sqrt{\Omega_r}} \int_z^\infty \frac{dz}{(1 + z)^3} = \frac{1}{H_0 \sqrt{\Omega_r}} \frac{1}{2(1 + z)^2} = \frac{1}{2H(z)} \tag{1}
\]

and

\[
r_H(z) = \int_z^\infty \frac{c \, dz}{H(z)} = \frac{c}{H_0 \sqrt{\Omega_r}} \int_z^\infty \frac{dz}{(1 + z)^2} = \frac{c}{H_0 \sqrt{\Omega_r}} \frac{1}{(1 + z)} = \frac{c(1 + z)}{H(z)} = 2ct(z)(1 + z) \tag{2}
\]

The only differences between these formulae and our earlier formulae are the limits of integration.

Note the scalings of \( r_H(z) \): \( ct \) is the physical distance travelled, \( 1 + z \) corrects this to a comoving distance assuming that all of the travelling was done at the final redshift \( z \), and a factor of 2 corrects for the fact that travel at earlier times was slightly more efficient in terms of comoving distance.

Problem 2 (5 pts):  
a) Now we use \( H(z) = H_0 \left[ \Omega_r (1 + z)^4 + \Omega_m (1 + z)^3 \right]^{1/2} \). The integral can be done by changing variables to \( R = (1 + z)^{-1} \). \( dR = -dz/(1 + z)^2 \) or \( dz = dR/R^2 \). It is useful to define the epoch of matter-radiation equality as \( 1 + z_{eq} = \Omega_m/\Omega_r \).

For the comoving distance, we have

\[
r_H(z) = \int_z^\infty \frac{c \, dz}{H(z)} = \frac{c}{H_0 \sqrt{\Omega_m}} \int_0^{(1+z)^{-1}} \frac{dR}{R^2 \sqrt{\Omega_r R^{-4} + \Omega_m R^{-3}}} = \frac{c}{H_0 \sqrt{\Omega_m}} \int_0^{(1+z)^{-1}} \frac{dR}{\sqrt{\Omega_r + \Omega_m R}} \tag{3}
\]

\[
= \frac{2c}{H_0 \sqrt{\Omega_m}} \left[ \sqrt{\Omega_r + \Omega_m R} \right]_0^{(1+z)^{-1}} = \frac{2c}{H_0 \sqrt{\Omega_m}} \left[ \sqrt{\Omega_r} + \frac{1}{1+z} - \sqrt{\Omega_r} \right] \tag{4}
\]

\[
= \frac{2c}{H_0 \sqrt{\Omega_m \sqrt{1 + z_{eq}}}} \left[ \sqrt{1 + y} - 1 \right] \tag{5}
\]

where \( y = (1 + z_{eq})/(1 + z) \propto R \).

For early times with \( y \ll 1 \), the term in square brackets is \( y/2 \), which yields

\[
r_H(z) \to \frac{c \sqrt{1 + z_{eq}}}{H_0 \sqrt{\Omega_m (1 + z)}} = \frac{c(1 + z)}{H_0 \sqrt{\Omega_r (1 + z)^2}} = \frac{c(1 + z)}{H(z)} \tag{6}
\]

as in Problem 1.

b) We use \( \Omega_r h^2 = 4.2 \times 10^{-5} \) and \( \Omega_m h^2 = 0.147 \), so \( 1 + z_{eq} = 3500 \). At \( z = 1000 \), we have \( y = 3.5 \), which makes \( r_H = 0.069c/H_0 = 208h^{-1} \) Mpc.

At \( z = 3500 \), we have \( y = 1 \), which makes \( r_H = 0.026c/H_0 = 77h^{-1} \) Mpc.
Obviously these comoving distances are much smaller than the current size of the observable universe!

**Problem 3 (5 pts):** The optical depth can be computed as an integral along the line of sight of the cross-section times the density of free electrons.

\[
\tau = \int d\ell \sigma_T n_e = \int dt c\sigma_T n_e = \int \frac{c \, dz}{(1+z)H(z)} \sigma_T n_e \tag{7}
\]

We will use \( H(z) = H_0 \sqrt{\Omega_m (1+z)^3} \). I had only required \( \Omega_m = 1 \), but this form allows us to compute the optical depth at \( z \gg 1 \) for other cosmologies.

The number density of free electrons will scale as \((1+z)^3\). The present day value is

\[
n_{e,0} \approx \frac{\rho_b}{m_p} = \frac{\Omega_b \rho_{\text{crit}}}{m_p} = \Omega_b h^2 \frac{1.88 \times 10^{-29} \text{ g cm}^{-3}}{m_p} = 1.12 \times 10^{-5} \Omega_b h^2 \text{ cm}^{-3} \tag{8}
\]

where \( m_p \) is the mass of the proton.

The optical depth is then

\[
\tau = \int_0^z \frac{d\tau}{H_0 \sqrt{\Omega_m (1+z)^{5/2}}} = \frac{2c\sigma_T n_{e,0} (1+z)^{3/2}}{3H_0 \sqrt{\Omega_m}} \left[ (1+z)^{3/2} - 1 \right] \tag{9}
\]

\[
= 9.2 \times 10^{-4} h^{-1} (1+z)^{3/2} \Omega_m^{-1/2} \left( \frac{\Omega_b h^2}{0.02} \right) \tag{10}
\]

where the last line assumes \( z \gg 1 \) (consistent with our \( H(z) \) approximation).

If \( h = 0.7 \) and \( \Omega_m = 1 \), then \( \tau \) will reach unity at \( z = 82 \).

In fact, we expect that the universe becomes reionized at \( z = 10 \) or 20. For such redshifts, the optical depth is substantially less than 1, but it is still noticeably non-zero (\( \tau \approx 0.15 \)). The WMAP satellite claims to see a signature of this optical depth!

**Problem 4 (5 pts):** a) If \( X \) decouples when the universe is much hotter than \( m_X c^2 \), then it will be in an ultrarelativistic thermal distribution. Since \( X \) is a boson and there are two spin states (\( X \) and \( \bar{X} \)), the number density will be the same as the photons. Today, that is \( 411 \text{ cm}^{-3} \).

\[
\Omega_X = \frac{\rho_X}{\rho_c} = m_X n_X / \rho_c, \quad \text{so} \quad \Omega_X h^2 = 4 \times 10^7 (m_X c^2 / 1 \text{ GeV})! \] This is vastly more than is observed.

An additional (and optional) refinement is to include the extra heating of the photons since the universe had a temperature of \( \gg 1 \text{ GeV} \). We know that there is a factor of \((4/11)^{1/3} \) getting back to a few MeV. At that time, \( g_* = 10.75 \), 2 for photons, \( 2 \times 2 \times (7/8) \) for electrons, and \( 6 \times 1 \times (7/8) \) for neutrinos. At \( \gg 1 \text{ GeV} \), we would have additional factors of \( 4(7/8) \) each for the muon and tau leptons, 8 for gluons, and \( 2 \times 2 \times 3 \times (7/8) \) for each quark (say, 5 neglecting the top quark; here we have 2 spins and 3 colors plus antiquarks), so \( g_* \approx 80 \). Hence, the annihilation of these species between 1 MeV and 10 GeV would increase the temperature at late times by another factor of \( \sim 8^{1/3} \). Taking these together, we would predict that the number of \( X \) particles is suppressed by a factor of 20. At \( \gg 100 \text{ GeV} \), we would have additional terms for the \( W \) and \( Z \) bosons, the top quark, and perhaps the Higgs sector, further reducing the \( X \) abundance.