Solutions for Assignment 4
Astronomy 541

Problem 1 (5 pts): The potential satisfies the Poisson equation \( \nabla^2 \phi = 4\pi G \rho \delta \), where \( \nabla_r \) is a derivative with respect to comoving coordinates. For matter, \( \rho \propto R^{-3} \). The growing mode scales as \( D(t) \). In \( \Omega_m = 1 \), \( D \propto R \). Hence, the right-hand side of the Poisson equation is time-independent. In linear perturbation theory, the spatial dependences of the perturbation are unchanging, so gradient has no time dependence. Hence, \( \phi \) is constant.

In \( \Omega_m < 1 \), \( D \) grows slower than \( R \). The RHS of the equation then declines in time (as \( R^{-1} \) if \( D \to \) constant). Hence, \( \phi \) decays in time, eventually as \( R^{-1} \).

Problem 2 (5 pts): a) For \( \Omega_m = 0.3 \), we have \( x = 7/3 \) at \( z = 0 \), \( x = 7/6 \) at \( z = 1 \), and \( x = 7/300 \) at \( z = 99 \). Using the formula, \( D = 0.00921 \) at \( z = 99 \), \( D = 0.288 \) at \( z = 1 \), and \( D = 0.426 \) at \( z = 0 \). From \( z = 99 \), amplitudes have grown by a factor of \( 46.3 \), compared to \( 100 \) in \( \Omega = 1 \). From \( z = 1 \), amplitudes have grown by a factor of \( 1.48 \), compared to \( 2 \) in \( \Omega = 1 \).

b) Here we simply use \( z_i = 99 \) and compute \( D(0)/D(99) = 77.8 \).

We didn’t ask you to compute \( D(0)/D(1) \) in this case. Should you want to try it, note that redshift 1 is not high enough to simply insert \( z_i = 1 \). Instead, you have keep a very high \( z_i \) and rescale all of the \( \Omega \)'s and \( z_i \) to what an observer at \( z = 1 \) would see. As \( z = 1 \), an astronomer would measure \( \Omega_m(z) = 0.774 \) and \( \Omega_A(z) = 0.226 \) and would claim that what was redshift 99 for a \( z = 0 \) observer was actually at \( z_i = 49 \). Putting these together, we find that the growth from \( z = 99 \) to \( z = 1 \) is 47.7 and hence that \( D(0)/D(1) = 1.63 \).

Problem 3 (10 pts): a) \( P \) has the dimensions of volume (comoving volume if you’ve labeled the position of the particles in comoving coordinates, as you would in a linear theory calculation). Hence, \( A \) has units of length\(^4 \).

b) \( P \) is maximized where \( dP/dk = 0 \). This is at \( k = 1/s\sqrt{3} \).

c) \( \sigma^2_K = \int K^2 \frac{k^2 dk}{2\pi^2} P(k) \). Let \( x = k^2 s^2 \). Then

\[
\sigma^2_K = \frac{A}{4\pi^2 s^4} \int_0 K^2 s^2 \frac{x dx}{(1 + x)^2} = \frac{A}{4\pi^2 s^4} \left[ \ln(1 + x) + \frac{1}{1 + x} \right]_0^{K^2 s^2} = \frac{A}{4\pi^2 s^4} \left[ \ln(1 + K^2 s^2) - \frac{1}{1 + (Ks)^2} \right]
\]

We can approximate \( \sigma_R \) by using \( K = \pi/2R \). Note that this \( R \) is just a chosen radius, not the scale factor of the universe.

\( \sigma_K \) goes as \( K^2 \) at small \( K \) and \( \ln K \) at large \( K \). There is (of course) no maximum at \( K \approx 1/s \), and the transition between the two asymptotic regimes is fairly slow.

d) For \( R = 8h^{-1} \) Mpc and \( s = 20h^{-1} \) Mpc, \( \sigma^2 = 0.096A s^{-4} \). To make \( \sigma = 0.9 \), we must have \( A = 17.20s^4 \). Then \( \sigma = 0.065 \) at \( R = 80h^{-1} \) Mpc and \( \sigma = 1.66 \) at \( R = 0.8h^{-1} \) Mpc.

e) \( M = \left(4\pi/3\right)\rho_c \Omega_m R^3 \). \( \rho_c = 278h^2 \times 10^9 \text{ M}_\odot \text{ Mpc}^{-3} \). For \( \Omega_m = 0.3 \), \( R = 8h^{-1} \) Mpc defines a mass of \( M = 1.79h^{-1} \times 10^{14} \text{ M}_\odot \), which is comparable to a cluster of galaxies (but not the richest ones). \( R = 0.8h^{-1} \) Mpc is 1000 times smaller, which is a galaxy size, albeit about a factor of
10 below a $L^*$ galaxy like the Milky Way. $R = 80h^{-1}$ Mpc is 1000 bigger, well larger than any collapsed structure and even the recognized superclusters.